Improved bounds for the sunflower lemma

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Definitions

Definition \((w\text{-set system and } r\text{-sunflower})\)

A \(w\)-set system is a family of sets of size at most \(w\).
Definitions

Definition (\(w\)-set system and \(r\)-sunflower)

A \(w\)-set system is a family of sets of size at most \(w\).

An \(r\)-sunflower is \(r\) sets \(S_1, \ldots, S_r\) where

- **Kernel**: \(Y = S_1 \cap \ldots \cap S_r\);
- **Petals**: \(S_1 \setminus Y, \ldots, S_r \setminus Y\) are pairwise disjoint.
Definitions

Definition ($w$-set system and $r$-sunflower)

A $w$-set system is a family of sets of size at most $w$. An $r$-sunflower is $r$ sets $S_1, \ldots, S_r$ where

- **Kernel**: $Y = S_1 \cap \cdots \cap S_r$;
- **Petals**: $S_1 \setminus Y, \ldots, S_r \setminus Y$ are pairwise disjoint.

Example

$\{\{1, 2\}, \{1, 3, 4, 6\}, \{1, 5\}, \{2, 3\}\}$ is a 4-set system of size 4. It has a 3-sunflower $\{\{1, 2\}, \{1, 3, 4, 6\}, \{1, 5\}\}$ with kernel $\{1\}$ and petals $\{2\}, \{3, 4, 6\}, \{5\}$. 
Main result

**Theorem (Erdős-Rado sunflower)**

*Any* \( w \)-set system of size \( s \) *has an* \( r \)-sunflower.*
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*Any* $w$-*set system of size* $s$ *has an* $r$-*sunflower.*

Let’s focus on $r = 3$. 

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- Kostochka 2000: $s \approx (w \log \log \log w / \log \log w)^w$.  

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- Erdős and Rado 1960: $s = w! \cdot 2^w \approx w^w$.
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- Fukuyama 2018: $s \approx w^{0.75w}$.
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- Fukuyama 2018: $s \approx w^{0.75w}$.
- Now: $s \approx (\log w)^w$ and this is tight for our approach.
Actual bound and further refinement

Theorem (Improved sunflower lemma)

For some constant $C$, any $w$-set system of size $s$ has an $r$-sunflower, where

$$ s = \left( Cr^2 \cdot \left( \log w \log \log w + (\log r)^2 \right) \right)^w. $$
Actual bound and further refinement

**Theorem (Improved sunflower lemma)**

For some constant $C$, any $w$-set system of size $s$ has an $r$-sunflower, where

$$s = \left(Cr^2 \cdot (\log w \log \log w + (\log r)^2)\right)^w.$$

Recently, Anup Rao improved it to

$$s = \left(Cr(\log w + \log r)\right)^w.$$
Applications – Theoretical computer science

- Circuit lower bounds
- Data structure lower bounds
- Matrix multiplication
- Pseudorandomness
- Cryptography
- Property testing
- Fixed parameter complexity
- Communication complexity
- ...

Applications – Combinatorics

- Erdős-Szemerédi sunflower lemma
- Intersecting set systems
- Packing Kneser graphs
- Alon-Jaeger-Tarsi nowhere-zero conjecture
- Thresholds in random graphs
- ...

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**Section 3**

**Proof overview**
Make it robust

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ is a $w$-set system.
Make it robust

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ is a $w$-set system.
Define a width-$w$ DNF $f_\mathcal{F}$ as $f_\mathcal{F} = \bigvee_{i=1}^{m} \bigwedge_{j \in S_i} x_j$.

Example

If $\mathcal{F} = \{\{1, 2\}, \{1, 3, 4, 6\}, \{1, 5\}, \{2, 3\}\}$, then

$f_\mathcal{F} = (x_1 \land x_2) \lor (x_1 \land x_3 \land x_4 \land x_6) \lor (x_1 \land x_5) \lor (x_2 \land x_3)$. 
Make it robust

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ is a $w$-set system. Define a width-$w$ DNF $f_\mathcal{F}$ as $f_\mathcal{F} = \bigvee_{i=1}^{m} \bigwedge_{j \in S_i} x_j$.

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**Definition (Satisfying system)**

$\mathcal{F}$ is satisfying if $\Pr [f_\mathcal{F}(x) = 0] < 1/3$ with $\Pr [x_i = 1] = 1/3$, \[ \forall i \in \{1, 2, \ldots, m\}, S_i \not\subseteq S_{i+1}. \]
Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ is a $w$-set system.
Define a width-$w$ DNF $f_\mathcal{F}$ as $f_\mathcal{F} = \bigvee_{i=1}^{m} \bigwedge_{j \in S_i} x_j$.

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If $\mathcal{F} = \{\{1, 2\}, \{1, 3, 4, 6\}, \{1, 5\}, \{2, 3\}\}$, then
$f_\mathcal{F} = (x_1 \land x_2) \lor (x_1 \land x_3 \land x_4 \land x_6) \lor (x_1 \land x_5) \lor (x_2 \land x_3)$.

Definition (Satisfying system)

$\mathcal{F}$ is satisfying if $\Pr[f_\mathcal{F}(x) = 0] < 1/3$ with $\Pr[x_i = 1] = 1/3$,
i.e., $\Pr[\forall i \in [m], S_i \not\subset S] < 1/3$ with $\Pr[x_i \in S] = 1/3$. 
Satisfyingness implies sunflower

Assume $\mathcal{F}$ is a set system on ground set $\{1, \ldots, n\}$.

**Lemma**

*If $\mathcal{F}$ is satisfying, then it has 3 pairwise disjoint sets.*
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**Proof.**

Color $x_1, \ldots, x_n$ to red, green, blue uniformly and independently.
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Proof.

Color $x_1, \ldots, x_n$ to red, green, blue uniformly and independently. By definition, $\mathcal{F}$ contains a purely red (green/blue) set w.p. $> 2/3$. 
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*If $\mathcal{F}$ is satisfying, then it has 3 pairwise disjoint sets.*

3 pairwise disjoint sets is a 3-sunflower with empty kernel.

Proof.

Color $x_1, \ldots, x_n$ to red, green, blue uniformly and independently. By definition, $\mathcal{F}$ contains a purely red (green/blue) set w.p. $> 2/3$. By union bound, $\mathcal{F}$ contains one purely red set, one purely green set, and one purely blue set w.p. $> 0$.  

□
Structure vs pseudorandomness

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}, m > \kappa^w$ is a $w$-set system. Define link $\mathcal{F}_Y = \{S_i \setminus Y \mid Y \subset S_i\}$, which is a $(w - |Y|)$-set system.

Example

If $\mathcal{F} = \{\{1, 2\}, \{1, 3, 4\}, \{1, 5\}, \{2, 3\}\}$, then $\mathcal{F}_{\{2\}} = \{\{1\}, \{3\}\}$. 
Structure vs pseudorandomness

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$, $m > \kappa^w$ is a $w$-set system. Define link $\mathcal{F}_Y = \{S_i \setminus Y \mid Y \subset S_i\}$, which is a $(w - |Y|)$-set system.

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If there exists $Y$ such that $|\mathcal{F}_Y| \geq m/\kappa^{|Y|} > \kappa^{w-|Y|}$, then we can apply induction and find an 3-sunflower in $\mathcal{F}_Y$. 
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If there exists $Y$ such that $|\mathcal{F}_Y| \geq m/\kappa^{|Y|} > \kappa^{w-|Y|}$, then we can apply induction and find an 3-sunflower in $\mathcal{F}_Y$.

So induction starts at such $\mathcal{F}$, that $|\mathcal{F}_Y| < m/\kappa^{|Y|}$ holds for any $Y$.

Lemma

Let $\kappa \geq (\log w)^{O(1)}$. If $|\mathcal{F}_Y| < m/\kappa^{|Y|}$ holds for any $Y$, then $\mathcal{F}$ is satisfying, which means $\mathcal{F}$ has 3 pairwise disjoint sets.
Randomness preserves pseudorandomness

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a $w$-(multi-)set system.
Randomness preserves pseudorandomness

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a $w$-(multi-)set system. Assume $|\mathcal{F}_Y| < m/\kappa|Y|$ holds for any $Y$. $\iff \mathcal{F}$ is pseudorandom
Randomness preserves pseudorandomness

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a $w$-(multi-)set system. Assume $|\mathcal{F}_Y| < m/\kappa|Y|$ holds for any $Y$. ⇔ $\mathcal{F}$ is pseudorandom

Take $\approx 1/\sqrt{\kappa}$-fraction of the ground set as $W$, and construct a $w/2$-(multi-)set system $\mathcal{F}'$ from each $S_i$: 
Randomness preserves pseudorandomness

Let \( \mathcal{F} = \{S_1, \ldots, S_m\} \) be a \( w \)-\( ( \text{multi-} ) \)set system. Assume \(|\mathcal{F}_Y| < \frac{m}{\kappa}|Y|\) holds for any \( Y \). \( \Leftarrow \) \( \mathcal{F} \) is pseudorandom

Take \( \approx \frac{1}{\sqrt{\kappa}} \)-fraction of the ground set as \( W \), and construct a \( \frac{w}{2} \)-\( ( \text{multi-} ) \)set system \( \mathcal{F}' \) from each \( S_i \):

- **Good:** If there exists \( |S_j \setminus W| \leq \frac{w}{2} \) and \( S_j \setminus W \subseteq S_i \setminus W \), then put \( S_j \setminus W \) into \( \mathcal{F}' \); (\( j \) may equal \( i \))
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Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a $w$-(multi-)set system. Assume $|\mathcal{F}_Y| < m/\kappa|Y|$ holds for any $Y$. ⇐ $\mathcal{F}$ is pseudorandom

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  E.g., $S_j \setminus W = \{1\}, S_i \setminus W = \{1, 2, 3, 4, 5\}$. 
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  E.g., $S_j \setminus W = \{1\}$, $S_i \setminus W = \{1, 2, 3, 4, 5\}$.

- **Bad**: otherwise, we do nothing for $S_i$.

Example

If $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{4, 5, 6, 7\}\}$ and $w = 4$, $W = \{1\}$, then $\mathcal{F}' = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{4, 5, 6, 7\}\}$. 
One reduction step

Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\Leftarrow \mathcal{F}'$ is also pseudorandom
One reduction step

Then $|F'_Y| \leq |F_Y|$ and $|F'| \approx |F|$. $\iff F'$ is also pseudorandom

Prove by encoding bad $(W, i) \rightarrow (W' = W \cup S_i, \text{aux}_1, k, \text{aux}_2)$, where $S_i$ ranks $k < |F|/\kappa^{w/2}$ in $F_{S_j \cap S_i}$ for the first $j \leq i$ that $S_j \setminus W \subset S_i \setminus W$. 
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Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\iff \mathcal{F}'$ is also pseudorandom

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Example

$\mathcal{F}' = \{\{1,2\}, \{2,3,4\}, \{1,4,5,6\}, \{4,5,6,7\}\}, W = \{1\}, i = 4.$

Encode/decode bad pair $(W, i)$:
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Encode/decode bad pair $(W, i)$:

- $W' = W \cup S_i = \{1,4,5,6,7\}$ we find $j = 3$ with $S_j \subset W'$
One reduction step

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Prove by encoding bad $(W, i) \rightarrow (W' = W \cup S_i, aux_1, k, aux_2)$, where $S_i$ ranks $k < |F|/\kappa w^2$ in $F_{S_j \cap S_i}$ for the first $j \leq i$ that $S_j \setminus W \subset S_i \setminus W$.

Example

$F' = \{\{1,2\}, \{2,3,4\}, \{1,4,5,6\}, \{4,5,6,7\}\}$, $W = \{1\}$, $i = 4$.

Encode/decode bad pair $(W, i)$:

- $W' = W \cup S_i = \{1,4,5,6,7\}$ we find $j = 3$ with $S_j \subset W'$
- $aux_1 = *$$$ with at least $w/2$ $s$
One reduction step

Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. ⇐ $\mathcal{F}'$ is also pseudorandom

Prove by encoding bad $(W, i) \rightarrow (W' = W \cup S_i, \text{aux}_1, k, \text{aux}_2)$, where $S_i$ ranks $k < |\mathcal{F}|/\kappa w^2$ in $\mathcal{F}_{S_j \cap S_i}$ for the first $j \leq i$ that $S_j \setminus W \subset S_i \setminus W$.

Example

$\mathcal{F}' = \{ \{1,2\}, \{2,3,4\}, \{1,4,5,6\}, \{4,5,6,7\} \}, W = \{1\}, i = 4$. Encode/decode bad pair $(W, i)$:

- $W' = W \cup S_i = \{1,4,5,6,7\}$ we find $j = 3$ with $S_j \subset W'$
- $\text{aux}_1 = *$$*$$*$$*$$*$$*$$*$$*$$*$$*$$*$$*$$*$$*$ with at least $w/2$ $\$s we know $S_j \cap S_i = \{4,5,6\}$
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Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\Leftarrow \mathcal{F}'$ is also pseudorandom
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Example

$\mathcal{F}' = \{\{1,2\}, \{2,3,4\}, \{1,4,5,6\}, \{4,5,6,7\}\}, W = \{1\}, i = 4$. Encode/decode bad pair $(W, i)$:

- $W' = W \cup S_i = \{1,4,5,6,7\}$ we find $j = 3$ with $S_j \subset W'$
- $\text{aux}_1 = *$$\$$ with at least $w/2$ $\$$s we know $S_j \cap S_i = \{4,5,6\}$
- $k = 2$
One reduction step

Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\iff$ $\mathcal{F}'$ is also pseudorandom

Prove by encoding bad $(W, i) \rightarrow (W' = W \cup S_i, \text{aux}_1, k, \text{aux}_2)$, where $S_i$ ranks $k < |\mathcal{F}|/\kappa w^2$ in $\mathcal{F}_{S_j \cap S_i}$ for the first $j \leq i$ that $S_j \setminus W \subset S_i \setminus W$.

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- $k = 2$ $S_i$ ranks 2 in $\mathcal{F}_{\{4,5,6\}}$, we recover $i = 4$
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Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\Leftarrow \mathcal{F}'$ is also pseudorandom

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Example

$\mathcal{F}' = \{(1,2), (2,3,4), (1,4,5,6), (4,5,6,7)\}, W = \{1\}, i = 4.$

Encode/decode bad pair $(W, i)$:

- $W' = W \cup S_i = \{1, 4, 5, 6, 7\}$ we find $j = 3$ with $S_j \subset W'$
- $\text{aux}_1 = \ast\ast\ast\ast$ with at least $w/2$ $\$$s we know $S_j \cap S_i = \{4, 5, 6\}$
- $k = 2$ $S_i$ ranks 2 in $\mathcal{F}_{\{4,5,6\}}$, we recover $i = 4$
- $\text{aux}_2 = \ast\ast\ast\ast$
One reduction step

Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\Leftarrow$ $\mathcal{F}'$ is also pseudorandom

Prove by encoding bad $(W, i) \rightarrow (W' = W \cup S_i, \text{aux}_1, k, \text{aux}_2)$, where $S_i$ ranks $k < |\mathcal{F}|/\kappa w^2$ in $\mathcal{F}_{S_j \cap S_i}$ for the first $j \leq i$ that $S_j \setminus W \subset S_i \setminus W$.

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- $\text{aux}_1 = *$$$ with at least $w/2$ $\$$s we know $S_j \cap S_i = \{4, 5, 6\}$
- $k = 2$ $S_i$ ranks 2 in $\mathcal{F}_{\{4, 5, 6\}}$, we recover $i = 4$
- $\text{aux}_2 = $$$ we recover $W = W' \setminus \{4, 5, 6, 7\}$
Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a $w$-(multi-)set system on $\{1, \ldots, n\}$. Assume $|\mathcal{F}_Y| < m/\kappa|Y|$ holds for any $Y$, and $\kappa \approx (\log w)^2$. 

Reductions
Let \( \mathcal{F} = \{S_1, \ldots, S_m\} \) be a \( w \)-\( (\text{multi-}) \)set system on \( \{1, \ldots, n\} \).
Assume \( |\mathcal{F}_Y| < m/\kappa |Y| \) holds for any \( Y \), and \( \kappa \approx (\log w)^2 \).
It suffices to prove

- \( \mathcal{F} \) is satisfying

\[ \iff \] w.h.p \( \mathcal{S} \) covers some set of \( \mathcal{F} \), where \( \Pr [x_i \in \mathcal{S}] = 1/3 \).
Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a $w$-(multi-)set system on $\{1, \ldots, n\}$. Assume $|\mathcal{F}_Y| < m/\kappa |Y|$ holds for any $Y$, and $\kappa \approx (\log w)^2$. It suffices to prove

- $\mathcal{F}$ is satisfying
  \[ \iff \text{w.h.p } S \text{ covers some set of } \mathcal{F}, \text{ where } \Pr \left[ x_i \in S \right] = 1/3. \]

Split $S$ into several parts,

- $\Pr \left[ x_i \in S \right] = 1/3$
  - $\approx$ take $1/3$-fraction of the ground set as $S$
  - $\approx$ view $S$ as $W_1, W_2, \ldots, W_{\log w}$, each of $\approx 1/\sqrt{\kappa}$-fraction

Then we iteratively apply (pseudorandom-preserving) reductions,

\[ \mathcal{F} \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow \cdots \rightarrow \mathcal{F}_{\text{last}}. \]
Let $\mathcal{F} = \{ S_1, \ldots, S_m \}$ be a $w$-(multi-)set system on $\{1, \ldots, n\}$. Assume $|\mathcal{F}_Y| < m/\kappa |Y|$ holds for any $Y$, and $\kappa \approx (\log w)^2$. It suffices to prove

- $\mathcal{F}$ is satisfying
  \[ \iff \text{w.h.p} \ S \text{ covers some set of } \mathcal{F}, \text{ where } \Pr [x_i \in S] = 1/3.\]

Split $S$ into several parts,

- $\Pr [x_i \in S] = 1/3$
  \[ \approx \text{take } 1/3\text{-fraction of the ground set as } S \]
  \[ \approx \text{view } S \text{ as } W_1, W_2, \ldots, W_{\log w}, \text{ each of } \approx 1/\sqrt{\kappa}\text{-fraction}\]

Then we iteratively apply (pseudorandom-preserving) reductions,

\[ \mathcal{F} \xrightarrow{W_1} \mathcal{F}' \xrightarrow{W_2} \mathcal{F}'' \xrightarrow{W_3} \ldots \xrightarrow{W_{\log w}} \mathcal{F}\text{last}. \]
Final step

Recall $S = W_1 \cup \cdots \cup W_{\log w}$ and

$$ \mathcal{F} \text{ width-}w $$

either we stop at $W_i$ when some set is contained in $\bigcup_{j<i} W_j$, or we don’t stop. Then, $F_{\text{last}}$ is a width-0 (multi-)set system of size $\approx m/\kappa w$, and $|F_{\text{last}}| \leq |F_{\text{last}}|/\kappa |Y|$ still holds for any $Y$. Thus, (informally) we proved such $F$ is satisfying, which means $F$ has an $3$-sunflower (3 pairwise disjoint sets).
Final step

Recall $S = W_1 \cup \cdots \cup W_{\log w}$ and

$F \xrightarrow{W_1} F'$

width-$w$  width-$w/2$
Final step

Recall $S = W_1 \cup \cdots \cup W_{\log w}$ and

\[ \begin{array}{c}
\mathcal{F} \xrightarrow{W_1} \mathcal{F}' \xrightarrow{W_2} \mathcal{F}'' \\
\text{width-}w \quad \text{width-}w/2 \quad \text{width-}w/4
\end{array} \]
Final step

Recall $S = W_1 \cup \cdots \cup W_{\log w}$ and

$$
\begin{align*}
\mathcal{F} & \xrightarrow{W_1} \mathcal{F'} & \mathcal{F'} & \xrightarrow{W_2} \mathcal{F''} & \mathcal{F''} & \xrightarrow{W_3} \cdots \xrightarrow{W_{\log w}} \mathcal{F}_{\text{last}}.
\end{align*}
$$

\text{width-}w \quad \text{width-}w/2 \quad \text{width-}w/4 \quad \text{width-0}
Final step

Recall \( S = W_1 \cup \cdots \cup W_{\log w} \) and

\[
\begin{align*}
\mathcal{F} & \xrightarrow{W_1} \mathcal{F}' \xrightarrow{W_2} \mathcal{F}'' \xrightarrow{W_3} \cdots \xrightarrow{W_{\log w}} \mathcal{F}_{\text{last}}.
\end{align*}
\]

- either we stop at \( W_i \) when some set is contained in \( \bigcup_{j<i} W_j \),

\[ \Rightarrow S \text{ contains some set of } \mathcal{F} \]
Final step

Recall \( S = W_1 \cup \cdots \cup W_{\log w} \) and

\[
\begin{array}{cccccc}
\mathcal{F} & \xrightarrow{W_1} & \mathcal{F}' & \xrightarrow{W_2} & \mathcal{F}'' & \xrightarrow{W_3} \cdots \xrightarrow{W_{\log w}} \mathcal{F}^{\text{last}} \end{array}
\]

width-\( w \) width-\( w/2 \) width-\( w/4 \) width-\( 0 \)

- either we stop at \( W_i \) when some set is contained in \( \bigcup_{j<i} W_j \),
  \( \Rightarrow \) \( S \) contains some set of \( \mathcal{F} \)

- or we don’t stop.
  Then, \( \mathcal{F}^{\text{last}} \) is a width-0 (multi-)set system of size \( \approx m > \kappa^w \),
  and \( |\mathcal{F}_Y^{\text{last}}| \lesssim |\mathcal{F}^{\text{last}}| / \kappa^{|Y|} \) still holds for any \( Y \).
  \( \Rightarrow \) Impossible
Recall $S = W_1 \cup \cdots \cup W_{\log w}$ and

\[
F \quad W_1 \rightarrow \quad F' \quad W_2 \rightarrow \quad F'' \quad W_3 \rightarrow \cdots \quad W_{\log w} \rightarrow F^{\text{last}}.
\]

- either we stop at $W_i$ when some set is contained in $\bigcup_{j<i} W_j$, \Rightarrow $S$ contains some set of $F$

- or we don’t stop.

Then, $F^{\text{last}}$ is a width-0 (multi-)set system of size $\approx m > \kappa^w$, and $|F^{\text{last}}_Y| \leq |F^{\text{last}}| / \kappa^{|Y|}$ still holds for any $Y$.

\Rightarrow Impossible

Thus, (informally) we proved such $F$ is satisfying, which means $F$ has an 3-sunflower (3 pairwise disjoint sets).
Section 4

Open problems
Erdős-Rado sunflower

Problem (Erdős-Rado sunflower conjecture)

Any $w$-set system of size $O_r(1)^w$ has an $r$-sunflower.
Problem (Erdős-Rado sunflower conjecture)

Any $w$-set system of size $O_r(1)^w$ has an $r$-sunflower.

- Our approach cannot go beyond $(\log w)^{(1-o(1))w}$. 
Erdős-Rado sunflower

Problem (Erdős-Rado sunflower conjecture)

Any \( w \)-set system of size \( O_r(1)^w \) has an \( r \)-sunflower.

- Our approach cannot go beyond \( (\log w)^{(1-o(1))w} \).
- Lift the sunflower size?
  \[ r = 3 \implies r = 4. \]
Erdős-Rado sunflower

**Problem (Erdős-Rado sunflower conjecture)**

*Any* $w$-set system of size $O_r(1)^w$ *has an* $r$-sunflower.*

- Our approach cannot go beyond $(\log w)^{(1-o(1))w}$.
- Lift the sunflower size?
  - $r = 3 \iff r = 4$.
- Is $(\log w)^{(1-o(1))w}$ actually tight? Counterexamples?
Erdős-Szemerédi sunflower

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ and $S_i \subset \{1, 2, \ldots, n\}$.

**Problem (Erdős-Szemerédi sunflower conjecture)**

*There exists function $\varepsilon = \varepsilon(r) > 0$, such that, if $m > 2^n(1-\varepsilon)$, then $\mathcal{F}$ has an $r$-sunflower.*
Erdős-Szemerédi sunflower

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ and $S_i \subset \{1, 2, \ldots, n\}$.

**Problem (Erdős-Szemerédi sunflower conjecture)**

There exists function $\varepsilon = \varepsilon(r) > 0$, such that, if $m > 2^n(1-\varepsilon)$, then $\mathcal{F}$ has an $r$-sunflower.

- ER sunflower conjecture $\implies$ ES sunflower conjecture.
- Now:
  - general $r$: $\varepsilon = O_r \left( \frac{1}{\log n} \right)$ from ER sunflower.
  - $r = 3$: Naslund proved it using polynomial method.
Section 5

Thanks