Improved bounds for the sunflower lemma

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Joint work with

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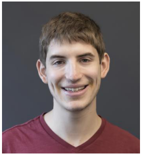

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Example

$\{\{1,2\}, \{1,3,4,6\}, \{1,5\}, \{2,3\}\}$ is a 4-set system of size 4.
It has a 3-sunflower $\{\{1,2\}, \{1,3,4,6\}, \{1,5\}\}$ with kernel $\{1\}$ and petals $\{2\}, \{3,4,6\}, \{5\}$.

Main result

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- Fukuyama 2018: $s \approx w^{0.75w}$.
- Now: $s \approx (\log w)^w$ and this is tight for our approach.

Actual bound and further refinement

Theorem (Improved sunflower lemma)

For some constant C, any w-set system of size s has an r-sunflower, where

$$s = \left(Cr^2 \cdot \left(\log w \log\log w + (\log r)^2\right)\right)^w .$$

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Recently, Anup Rao improved it to

$$s = \left(Cr(\log w + \log r)\right)^w .$$

Applications – Theoretical computer science

- Circuit lower bounds
- Data structure lower bounds
- Matrix multiplication
- Pseudorandomness
- Cryptography
- Property testing
- Fixed parameter complexity
- Communication complexity
- ...

Applications – Combinatorics

- Erdős-Szemerédi sunflower lemma
- Intersecting set systems
- Packing Kneser graphs
- Alon-Jaeger-Tarsi nowhere-zero conjecture
- Thersholds in random graphs
- ...

Section 3

Proof overview

Make it robust

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If $\mathcal{F} = \{\{1,2\}, \{1,3,4,6\}, \{1,5\}, \{2,3\}\}$, then
$f_{\mathcal{F}} = (x_1 \wedge x_2) \vee (x_1 \wedge x_3 \wedge x_4 \wedge x_6) \vee (x_1 \wedge x_5) \vee (x_2 \wedge x_3)$.

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i.e., $\Pr[\forall i \in [m], S_i \not\subset S] < 1/3$ with $\Pr[x_i \in S] = 1/3$.

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Assume $\mathcal{F}$ is a set system on ground set $\{1, \ldots, n\}$.

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By union bound, $\mathcal{F}$ contains one purely red set, one purely green set, and one purely blue set w.p > 0. $\square$

Structure vs pseudorandomness

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}, m > \kappa^w$ is a w-set system. Define link $\mathcal{F}_Y = \{S_i \backslash Y \mid Y \subset S_i\}$, which is a $(w - |Y|)$-set system.

Example

If $\mathcal{F} = \{\{1, 2\}, \{1, 3, 4\}, \{1, 5\}, \{2, 3\}\}$, then $\mathcal{F}_{\{2\}} = \{\{1\}, \{3\}\}$.

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So induction starts at such $\mathcal{F}$, that $|\mathcal{F}_Y| < m/\kappa^{|Y|}$ holds for any Y.

Lemma

Let $\kappa \geq (\log w)^{O(1)}$. If $|\mathcal{F}_Y| < m/\kappa^{|Y|}$ holds for any Y, then $\mathcal{F}$ is satisfying, which means $\mathcal{F}$ has 3 pairwise disjoint sets.

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  E.g., $S_j \backslash W = \{1\}, S_i \backslash W = \{1, 2, 3, 4, 5\}$.
- Bad: otherwise, we do nothing for S_i.

Example

If $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{4, 5, 6, 7\}\}$ and $w = 4, W = \{1\}$,
then $\mathcal{F}' = \{\{1,2\}, \{1,3\}, \{2, 3, 4\}, \{4, 5, 6, 7\}\}$.

One reduction step

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- $\mathsf{aux}_2 = \$\$\$\$$ we recover $W = W' \backslash \{4,5,6,7\}$

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Split S into several parts,

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Then we iteratively apply (pseudorandom-preserving) reductions,

$$\mathcal{F} \xrightarrow{W_1} \mathcal{F}' \xrightarrow{W_2} \mathcal{F}'' \xrightarrow{W_3} \dots \xrightarrow{W_{\log w}} \mathcal{F}^{\text{last}}.$$

Final step

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$$\underbrace{\mathcal{F}}_{\text{width-}w} \xrightarrow{W_1} \underbrace{\mathcal{F}'}_{\text{width-}w/2} \xrightarrow{W_2} \underbrace{\mathcal{F}''}_{\text{width-}w/4} \xrightarrow{W_3} \cdots \xrightarrow{W_{\log w}} \underbrace{\mathcal{F}^{\text{last}}}_{\text{width-}0} .$$

- either we stop at W_i when some set is contained in $\bigcup_{j<i} W_j$,
  $\Rightarrow S$ contains some set of $\mathcal{F}$

Final step

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- or we don't stop.
  Then, $\mathcal{F}^{\text{last}}$ is a width-0 (multi-)set system of size $\approx m > \kappa^w$,
  and $\left|\mathcal{F}^{\text{last}}_Y\right| \lessapprox \left|\mathcal{F}^{\text{last}}\right| / \kappa^{|Y|}$ still holds for any Y.
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Final step

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  $\Rightarrow$ Impossible

Thus, (informally) we proved such $\mathcal{F}$ is satisfying, which means $\mathcal{F}$ has an 3-sunflower (3 pairwise disjoint sets).

Section 4

Open problems

Erdős-Rado sunflower

Problem (Erdős-Rado sunflower conjecture)

Any w-set system of size $O_r(1)^w$ has an r-sunflower.

Erdős-Rado sunflower

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  $r = 3 \implies r = 4$.

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- Lift the sunflower size?
  $r = 3 \implies r = 4$.
- Is $(\log w)^{(1-o(1))w}$ actually tight? Counterexamples?

Erdős-Szemerédi sunflower

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ and $S_i \subset \{1, 2. \ldots, n\}$.

Problem (Erdős-Szemerédi sunflower conjecture)

There exists function $\varepsilon = \varepsilon(r) > 0$, such that, if $m > 2^{n(1-\varepsilon)}$, then $\mathcal{F}$ has an r-sunflower.

Erdős-Szemerédi sunflower

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Problem (Erdős-Szemerédi sunflower conjecture)

There exists function $\varepsilon = \varepsilon(r) > 0$, such that, if $m > 2^{n(1-\varepsilon)}$, then $\mathcal{F}$ has an r-sunflower.

- ER sunflower conjecture $\implies$ ES sunflower conjecture.
- Now:
  - general r: $\varepsilon = O_r\left(1/\log n\right)$ from ER sunflower.
  - $r = 3$: Naslund proved it using polynomial method.

Section 5

Thanks