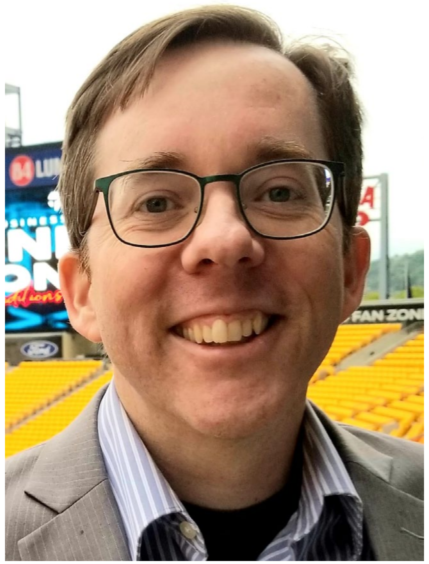
No exponential quantum speedup for SIS^{∞} anymore

Kewen Wu (IAS)



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Outline

Toy example: $\mathbf{F}_3^n$-Subset-Sum

Motivations

Main problem: the SIS^∞ problem

Cryptographic motivation

Full generalization: the $\boldsymbol{A}$-SIS problem

Quantum motivation

Algorithm overview

A toy SIS^{∞} problem

Given vectors in $\mathbf{F}_3^n$, *efficiently* find a nonempty subset of them that sums to zero

$\mathbf{F}_3^n$-Subset-Sum

A toy SIS^{∞} problem

Given vectors in $\mathbf{F}_3^n$, *efficiently* find a nonempty subset of them that sums to zero

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_3^n$

Output: $\emptyset \neq \boldsymbol{S} \subseteq [\boldsymbol{m}]$ or $\perp$

Condition: $\sum_{i \in S} \boldsymbol{v_i} \equiv \vec{\mathbf{0}} \bmod \mathbf{3}$ or no such $\boldsymbol{S}$

A toy SIS$^\infty$ problem

Given vectors in $\mathbf{F}_3^n$, *efficiently* find a nonempty subset of them that sums to zero

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Only allow $\mathbf{0}, \mathbf{1}$ as coefficients
$\mathbf{2}$ is not allowed

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_3^n$

Output: $\emptyset \neq \boldsymbol{S} \subseteq [\boldsymbol{m}]$ or $\perp$ such that $\sum_{i \in S} \boldsymbol{v_i} \equiv \vec{\mathbf{0}} \ \mathbf{mod}\ \mathbf{3}$

$$\boldsymbol{n} = \mathbf{4} \left\{ \begin{matrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} \end{matrix} \right.$$

$$\underbrace{\hphantom{\begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}}}_{\boldsymbol{m} = \mathbf{6}}$$

$\mathbf{F}_3^n$-Subset-Sum

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Output: $\emptyset \neq S \subseteq [m]$ or $\perp$ such that $\sum_{i \in S} v_i \equiv \vec{0} \bmod 3$

$$n = 4 \left\{ \vphantom{\begin{bmatrix}0\\2\\1\\2\end{bmatrix}} \right. \underbrace{\begin{bmatrix}0\\2\\1\\2\end{bmatrix} \quad \begin{bmatrix}1\\2\\1\\1\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\2\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\1\\2\end{bmatrix} \quad \begin{bmatrix}2\\2\\2\\1\end{bmatrix}}_{m = 6}$$

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$\mathbf{F}_3^n$-Subset-Sum landscape

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$

Output: $\emptyset \neq S \subseteq [m]$ or $\perp$ such that $\sum_{i \in S} v_i \equiv \vec{0} \bmod 3$

Harder *Easier*

$\longrightarrow \boldsymbol{m}$

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Harder *Easier*

n $\longrightarrow$ m

NP-hard

$\mathbb{F}_3^n$-Subset-Sum landscape

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Harder *Easier*

n $2n$ m

NP-hard

Total-Search

(no $\perp$)

$\mathbf{F}_3^n$-Subset-Sum landscape

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$

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Harder *Easier*

n $n + \sqrt{n}$ $2n$ $\longrightarrow m$

NP-hard

Learning-With-Error
(Arora-Ge'11)

Total-Search
(no $\perp$)

$\mathbb{F}_3^n$-Subset-Sum landscape

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$

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Harder *Easier*

n — **NP-hard**

$n + \sqrt{n}$ — **Learning-With-Error** (Arora-Ge'11)

$2n$ — **Total-Search** (no $\perp$)

Cn^2 — **Quantum polytime** (Chen-Liu-Zhandry'21)

m

$\mathbb{F}_3^n$-Subset-Sum landscape

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Harder *Easier*

n — $n + \sqrt{n}$ — $2n$ — $n^2/2$ → m

NP-hard

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(Arora-Ge'11)

Total-Search
(no $\perp$)

Quantum polytime
(Chen-Liu-Zhandry'21)

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Harder → *Easier*

m

n	$n + \sqrt{n}$	$2n$	$n^2/2$
NP-hard	**Learning-With-Error** (Arora-Ge'11)	**Total-Search** (no $\perp$)	**Quantum polytime** (Chen-Liu-Zhandry'21) **Classical polytime** (Ivanyos-Sanselma-Santha'07)

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Harder *Easier*

n $\quad n + \sqrt{n}$ $\quad 2n$ $\quad n^2/3$ $\quad n^2/2$ $\quad m$

NP-hard

Learning-With-Error
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Total-Search
(no $\perp$)

Classical polytime
(new)

Quantum polytime
(Chen-Liu-Zhandry'21)
Classical polytime
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Harder *Easier*

?

$\underbrace{2n \quad n^2/3}$

n	$n + \sqrt{n}$	$2n$	$n^2/3$	$n^2/2$	m
NP-hard	**Learning-With-Error** (Arora-Ge'11)	**Total-Search** (no $\perp$)	**Classical polytime (new)**	**Quantum polytime** (Chen-Liu-Zhandry'21) **Classical polytime** (Ivanyos-Sanselma-Santha'07)	

Why $\mathbf{F}_3^n$-Subset-Sum

Natural problem with many perspectives

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- Complexity theory: subset-sum and LIN-SAT

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Discrepancy theory: vector balancing over $\mathbf{F}_3$

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- Coding theory: ternary syndrome decoding with maximal weight

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Security of post-quantum signature scheme

- *Wave* (Debris-Alazard, Sendrier, Tillich'19)
- *CRYSTALS-Dilithium* (Ducas, Kiltz, Lepoint, Lyubashevsky, Schwabe, Seiler, Stehle'18)

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Quantum advantage

Why $\mathbf{F}_3^n$-Subset-Sum

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- Discrepancy theory: vector balancing over $\mathbf{F}_3$
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Quantum advantage

Warmup for the SIS^∞ problem

The SIS^{∞} problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \ldots, c_m}$ such that $\boldsymbol{\sum c_i v_i \equiv \vec{0} \bmod p}$

The SIS$^{\infty}$ problem

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{h \geq 1}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that each $|\boldsymbol{c_i}| \leq \boldsymbol{h}$ and $\sum \boldsymbol{c_i v_i} \equiv \vec{\mathbf{0}} \bmod \boldsymbol{p}$

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Short-**I**nteger-**S**olution

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Short-**I**nteger-**S**olution

Example.

Given $\boldsymbol{m}$ vectors in $\mathbf{F}_{101}^n$, find linear dependence where all coeffs $\boldsymbol{c_i}$ are between $\boldsymbol{\pm 5}$.

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Remark.

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- $\boldsymbol{p = 2}$ or $\mathbf{3}$: trivial

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- Smaller $\boldsymbol{h}$ is a harder problem
- $\boldsymbol{h = 1}$, allowing coeffs $\{\boldsymbol{-1, 0, +1}\}$: **Collision in linear hash**

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Remark.

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- $\boldsymbol{h} \geq \lfloor \boldsymbol{p}/\mathbf{2} \rfloor$: trivial

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- $\boldsymbol{h} = \mathbf{1}$, allowing coeffs $\{-\mathbf{1}, \mathbf{0}, +\mathbf{1}\}$: **Collision in linear hash**

$\boldsymbol{V} \in \mathbf{F}_p^{n \times m}$ defines a linear hash $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}^m \rightarrow \boldsymbol{Vx} \in \mathbf{F}_p^n$

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$\boldsymbol{Vx = Vx'}$ iff $\boldsymbol{V(x - x') = 0}$ iff $\boldsymbol{Vc = 0}$

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- $\boldsymbol{m \geq (p-1)n + 1}$: solution always exists

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Cryptography motivation.

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- Our focus: $\boldsymbol{m} \gg \boldsymbol{n}$, many solutions exist, find one

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Cryptography motivation.

- Our focus: $\boldsymbol{m} \gg \boldsymbol{n^2}$, many solutions exist, find one
- Learning-With-Error setting: $\boldsymbol{m = n + n^{1/C}}$, a (unique) solution *planted*, find it

The SIS^{∞} problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$ and $\boldsymbol{h \geq 1}$ where $\boldsymbol{p}$ is a prime

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- CRYSTALS-Dilithium signature scheme: in its *module* variant
  Assume $\boldsymbol{n = 1280, m = 2304, p \approx 2^{23}, h \approx p/8}$, random $\{\boldsymbol{v_i}\}$ is hard

The SIS^{∞} problem

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  Assume $\boldsymbol{n = 1280}$, $\boldsymbol{m = 2304}$, $\boldsymbol{p \approx 2^{23}}$, $\boldsymbol{h \approx p/8}$, random $\{\boldsymbol{v_i}\}$ is hard
  In general, assume $\boldsymbol{m \approx 1.9n}$, $\boldsymbol{p \gg n}$, $\boldsymbol{h \approx p/8}$, random $\{\boldsymbol{v_i}\}$ is hard

The SIS^∞ problem

Input: $\boldsymbol{v_1, \dots, v_m \in \mathbf{F}_p^n}$ and $\boldsymbol{h \geq 1}$ where $\boldsymbol{p}$ is a prime

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  Assume $\boldsymbol{n = 1280, m = 2304, p \approx 2^{23}, h \approx p/8}$, random $\{\boldsymbol{v_i}\}$ is hard
  In general, assume $\boldsymbol{m \approx 1.9n, p \gg n, h \approx p/8}$, random $\{\boldsymbol{v_i}\}$ is hard

$\boldsymbol{m \approx n \log n, p \approx n^2 \log n, h = O(1)}$, random $\{\boldsymbol{v_i}\}$ is hard,
based on worst-case hardness of lattice problems [Ajtai'96, Micciancio-Regev'04]

The SIS^∞ problem

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{h} \geq \mathbf{1}$ where $\boldsymbol{p}$ is a prime

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Theorem (Chen-Liu-Zhandry'21).
Assume $\boldsymbol{k}$ is odd constant and

$$\boldsymbol{m} \gg \boldsymbol{p}^{\mathbf{4}}\boldsymbol{n}^{\boldsymbol{k}}$$

Quantum polytime algorithm for

$$\boldsymbol{h} \geq \frac{\boldsymbol{p}-\boldsymbol{k}}{\mathbf{2}}$$

The SIS^{∞} problem

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{h \geq 1}$ where $\boldsymbol{p}$ is a prime

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Assume $\boldsymbol{k}$ is odd constant and

$$\boldsymbol{m \gg p^4 n^k}$$

Quantum polytime algorithm for

$$\boldsymbol{h \geq \frac{p-k}{2}}$$

Theorem (Imran-Ivanyos'25).
Assume $\boldsymbol{k}$ is a power of two and

$$\boldsymbol{m \gg p^{k \log k} n^k}$$

Classical polytime algorithm for

$$\boldsymbol{h \geq \frac{p}{2k}}$$

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$$\boldsymbol{m \gg p^4 n^k}$$

Quantum polytime algorithm for

$$\boldsymbol{h \geq \frac{p-k}{2}}$$

Theorem (Imran-Ivanyos'25).
Assume $\boldsymbol{k}$ is a power of two and

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Classical polytime algorithm for

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Theorem (New).
Assume $\boldsymbol{k}$ is any constant and

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Classical polytime algorithm for

$$\boldsymbol{h \geq \frac{p}{2k}}$$

The SIS$^\infty$ problem

Input: $\boldsymbol{v_1, \dots, v_m \in \mathbb{F}_p^n}$ and $\boldsymbol{h \geq 1}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that each $|\boldsymbol{c_i}| \leq \boldsymbol{h}$ and $\sum \boldsymbol{c_i v_i \equiv \vec{0} \bmod p}$

Theorem (Chen-Liu-Zhandry'21).
Assume $\boldsymbol{k}$ is odd constant and
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Quantum polytime algorithm for
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Runs in $\mathbf{poly}(\boldsymbol{n}, \mathbf{log}\, \boldsymbol{p})$ time

The SIS^∞ problem

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Classical polytime algorithm for

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Runs in $\mathbf{poly}(\boldsymbol{n}, \mathbf{log}\,\boldsymbol{p})$ time
Allows $\boldsymbol{p} = \mathbf{exp}(\mathbf{poly}(\boldsymbol{n}))$

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1, \dots, v_m \in \mathbf{F}_p^n}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that $\sum \boldsymbol{c_i v_i \equiv \vec{0} \bmod p}$

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1, \ldots, v_m \in \mathbf{F}_p^n}$ and $\boldsymbol{A \subseteq \mathbf{F}_p}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \ldots, c_m}$ such that each $\boldsymbol{c_i \in A}$ and $\boldsymbol{\sum c_i v_i \equiv \vec{0} \bmod p}$

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

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Example.

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1, \dots, v_m \in \mathbf{F}_p^n}$ and $\boldsymbol{A \subseteq \mathbf{F}_p}$ where $\boldsymbol{p}$ is a prime

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Example.

- $\boldsymbol{A = \{-h, -h+1, \dots, h-1, h\}}$ for $\mathbf{SIS}^{\infty}$

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that each $\boldsymbol{c_i} \in \boldsymbol{A}$ and $\sum \boldsymbol{c_i v_i} \equiv \vec{\mathbf{0}} \ \mathbf{mod}\ \boldsymbol{p}$

Example.

- $\boldsymbol{A} = \{-\boldsymbol{h}, -\boldsymbol{h} + \mathbf{1}, \dots, \boldsymbol{h} - \mathbf{1}, \boldsymbol{h}\}$ for $\mathbf{SIS}^{\infty}$
- $\boldsymbol{A} = \{\mathbf{0}, \mathbf{1}\}$ for $\mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$-Subset-Sum

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that each $\boldsymbol{c_i} \in \boldsymbol{A}$ and $\sum \boldsymbol{c_i v_i} \equiv \vec{\mathbf{0}} \mathbf{\ mod\ } \boldsymbol{p}$

Example.

- $\boldsymbol{A} = \{-\boldsymbol{h}, -\boldsymbol{h} + \mathbf{1}, \dots, \boldsymbol{h} - \mathbf{1}, \boldsymbol{h}\}$ for $\mathbf{SIS}^{\infty}$
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- $\boldsymbol{A} = \{-\mathbf{1}, \mathbf{0}, \mathbf{1}\}$ for Collision-Finding

General allowed set ($\boldsymbol{A}$-SIS)

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1}, \dots, \boldsymbol{c_m}$ such that each $\boldsymbol{c_i} \in \boldsymbol{A}$ and $\sum \boldsymbol{c_i} \boldsymbol{v_i} \equiv \vec{\mathbf{0}} \bmod \boldsymbol{p}$

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If $\mathbf{0} \notin \boldsymbol{A}$ and $\boldsymbol{v_1} = \boldsymbol{v_2} = \cdots = \boldsymbol{v_{m-1}} = \vec{\mathbf{0}}$ and $\boldsymbol{v_m} \neq \vec{\mathbf{0}}$, then it has no solution

General allowed set ($\boldsymbol{A}$-SIS)

Input: random $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that each $\boldsymbol{c_i} \in \boldsymbol{A}$ and $\sum \boldsymbol{c_i v_i} \equiv \vec{\mathbf{0}} \bmod \boldsymbol{p}$

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Theorem (Chen-Liu-Zhandry'21).
Assume $\mathbf{2} \leq \boldsymbol{k} \leq \boldsymbol{p} - \mathbf{1}$ is a constant and

$$\boldsymbol{m} \gg \boldsymbol{p}^{\mathbf{4}} \boldsymbol{n}^{\boldsymbol{k}}$$

Quantum polytime algorithm for $|\boldsymbol{A}| \geq \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$

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Input: random $\boldsymbol{v_1, \ldots, v_m \in \mathbf{F}_p^n}$ and $\boldsymbol{A \subseteq \mathbf{F}_p}$ where $\boldsymbol{p}$ is a prime

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Theorem (Chen-Liu-Zhandry'21).
Assume $\boldsymbol{2 \leq k \leq p-1}$ is a constant and

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Quantum polytime algorithm for $\boldsymbol{|A| \geq p-k+1}$

$\boldsymbol{k = 1}$: every coeff is allowed

General allowed set (A-SIS)

Input: random $v_1, \ldots, v_m \in \mathbf{F}_p^n$ and $A \subseteq \mathbf{F}_p$ where p is a prime

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Theorem (New).
Under the same $|A| \ge p-k+1$ condition
Classical polytime algorithm only needs

$$m \gg \log(p) \cdot \left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right.$$

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Quantum polytime algorithm for $|A| \ge p-k+1$

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$$m \gg \log(p) \cdot \begin{cases} n^2 & \text{if } p > 4^{k-1} \\ n^{k-1} & \text{if } p \ge 7 \text{ and } k \ge 3 \end{cases}$$

General allowed set (A-SIS)

Input: random $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$ where $\boldsymbol{p}$ is a prime

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Quantum polytime algorithm for $|\boldsymbol{A}| \geq \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$

$\boldsymbol{k} = \mathbf{1}$: every coeff is allowed
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Theorem (New).
Under the same $|\boldsymbol{A}| \geq \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$ condition
Classical polytime algorithm only needs

$$\boldsymbol{m} \gg \mathbf{log}(\boldsymbol{p}) \cdot \begin{cases} \boldsymbol{n}^{\mathbf{2}} & \text{if } \boldsymbol{p} > \mathbf{4}^{\boldsymbol{k}-\mathbf{1}} \\ \boldsymbol{n}^{\boldsymbol{k}-\mathbf{1}} & \text{if } \boldsymbol{p} \geq \mathbf{7} \text{ and } \boldsymbol{k} \geq \mathbf{3} \\ \boldsymbol{n}^{\boldsymbol{k}} & \text{in general for all } \boldsymbol{p} \geq \mathbf{3} \text{ and } \boldsymbol{k} \geq \mathbf{2} \end{cases}$$

General allowed set (A-SIS)

Input: random $v_1, \dots, v_m \in \mathbf{F}_p^n$ and $A \subseteq \mathbf{F}_p$ where p is a prime

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A full dequantization!

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Theorem (Chen-Liu-Zhandry'21).
Assume $2 \le k \le p-1$ is a constant and

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Quantum polytime algorithm for $|A| \ge p - k + 1$

$k = 1$: every coeff is allowed
$k = p$: only one coeff is allowed

Theorem (New).
Under the same $|A| \ge p - k + 1$ condition
Classical polytime algorithm

A full dequantization!

Why is this dequantization interesting?

$$m \gg \log(p) \cdot \begin{cases} n^2 & \text{if } p > 4^{k-1} \\ n^{k-1} & \text{if } p \ge 7 \text{ and } k \ge 3 \\ n^k & \text{in general for all } p \ge 3 \text{ and } k \ge 2 \end{cases}$$

On quantum speedup

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Find problems where quantum algorithms have exponential speedup over classical ones

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- Simulation of quantum systems
- Hidden subgroup problem (factoring, discrete log)

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- Chen-Liu-Zhandry'21
- Yamakawa-Zhandry'22
- (DQI team) Jordan, Shutty, Wootters, Zalcman, Schmidhuber, King, Isakov, Khattar, Babbush'24
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Captured by $\boldsymbol{A}$-SIS

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \dots, c_m}$ such that each $\boldsymbol{c_i} \in \boldsymbol{A}$ and $\sum \boldsymbol{c_i v_i} \equiv \vec{\mathbf{0}} \ \mathbf{mod}\ \boldsymbol{p}$

Chen-Liu-Zhandry'21	**Yamakawa-Zhandry'22**	**DQI'24, Chailloux-Tillich'24**

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Random $\{\boldsymbol{v_i}\}$
$\boldsymbol{A}$ arbitrary

Yamakawa-Zhandry'22

DQI'24, Chailloux-Tillich'24

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Chen-Liu-Zhandry'21

Random $\{\boldsymbol{v_i}\}$
$\boldsymbol{A}$ arbitrary
$\boldsymbol{p} = \mathbf{poly}(\boldsymbol{n})$
$|\boldsymbol{A}| = \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$
$\boldsymbol{m} \approx \boldsymbol{n}^{\boldsymbol{k}}$

Yamakawa-Zhandry'22

DQI'24, Chailloux-Tillich'24

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

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Yamakawa-Zhandry'22

$\{\boldsymbol{v_i}\}$ folded RS code
$\boldsymbol{A}$ random

DQI'24, Chailloux-Tillich'24

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Yamakawa-Zhandry'22

$\{\boldsymbol{v_i}\}$ folded RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{exp}(\boldsymbol{n} \log \boldsymbol{n})$
$|\boldsymbol{A}| = \boldsymbol{p}/\mathbf{2}$
$\boldsymbol{m} \approx \mathbf{6}\boldsymbol{n}$

DQI'24, Chailloux-Tillich'24

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

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DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v_i}\}$ RS code
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$\boldsymbol{p} = \mathbf{exp}(\boldsymbol{n} \log \boldsymbol{n})$
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$\boldsymbol{m} \approx \mathbf{6}\boldsymbol{n}$

DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v_i}\}$ RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{4}\boldsymbol{n}$
$|\boldsymbol{A}| = \boldsymbol{p}/\mathbf{2}$
$\boldsymbol{m} = \mathbf{4}\boldsymbol{n}$

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ such that each $\boldsymbol{c}_i \in \boldsymbol{A}$ and $\sum \boldsymbol{c}_i \boldsymbol{v}_i \equiv \vec{\mathbf{0}} \bmod \boldsymbol{p}$

Chen-Liu-Zhandry'21

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$\boldsymbol{A}$ arbitrary
$\boldsymbol{p} = \mathbf{poly}(\boldsymbol{n})$
$|\boldsymbol{A}| = \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$
$\boldsymbol{m} \approx \boldsymbol{n}^{\boldsymbol{k}}$

Now classically easy

Yamakawa-Zhandry'22

$\{\boldsymbol{v}_i\}$ folded RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{exp}(\boldsymbol{n} \log \boldsymbol{n})$
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DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v}_i\}$ RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{4}\boldsymbol{n}$
$|\boldsymbol{A}| = \boldsymbol{p}/\mathbf{2}$
$\boldsymbol{m} = \mathbf{4}\boldsymbol{n}$

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$ where $\boldsymbol{p}$ is a prime

Output: $\boldsymbol{c_1, \ldots, c_m}$ such that each $\boldsymbol{c_i} \in \boldsymbol{A}$ and $\sum \boldsymbol{c_i v_i} \equiv \vec{\mathbf{0}} \bmod \boldsymbol{p}$

Chen-Liu-Zhandry'21

Random $\{\boldsymbol{v_i}\}$
$\boldsymbol{A}$ arbitrary
$\boldsymbol{p} = \mathbf{poly}(\boldsymbol{n})$
$|\boldsymbol{A}| = \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$
$\boldsymbol{m} \approx \boldsymbol{n}^{\boldsymbol{k}}$

Now classically easy

Yamakawa-Zhandry'22

$\{\boldsymbol{v_i}\}$ folded RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{exp}(\boldsymbol{n} \log \boldsymbol{n})$
$|\boldsymbol{A}| = \boldsymbol{p}/\mathbf{2}$
$\boldsymbol{m} \approx \mathbf{6}\boldsymbol{n}$

Classically hard
in the query model

DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v_i}\}$ RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{4}\boldsymbol{n}$
$|\boldsymbol{A}| = \boldsymbol{p}/\mathbf{2}$
$\boldsymbol{m} = \mathbf{4}\boldsymbol{n}$

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$ where $\boldsymbol{p}$ is a prime

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Still seems
classically hard

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Why?

Chen-Liu-Zhandry'21

Random $\{\boldsymbol{v}_i\}$
$\boldsymbol{A}$ arbitrary
$\boldsymbol{p = \mathrm{poly}(n)}$
$\boldsymbol{|A| = p - k + 1}$
$\boldsymbol{m \approx n^k}$

Now classically easy

Yamakawa-Zhandry'22

$\{\boldsymbol{v}_i\}$ folded RS code
$\boldsymbol{A}$ random
$\boldsymbol{p = \exp(n \log n)}$
$\boldsymbol{|A| = p/2}$
$\boldsymbol{m \approx 6n}$

Classically hard
in the query model

DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v}_i\}$ RS code
$\boldsymbol{A}$ random
$\boldsymbol{p = 4n}$
$\boldsymbol{|A| = p/2}$
$\boldsymbol{m = 4n}$

Still seems
classically hard

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Why? We can handle worst-case $\{\boldsymbol{v_i}\}$, exponential $\boldsymbol{p}$, and $\boldsymbol{A}$ of large size

Chen-Liu-Zhandry'21

Random $\{\boldsymbol{v_i}\}$
$\boldsymbol{A}$ arbitrary
$\boldsymbol{p = \mathrm{poly}(n)}$
$\boldsymbol{|A| = p - k + 1}$
$\boldsymbol{m \approx n^k}$

Now classically easy

Yamakawa-Zhandry'22

$\{\boldsymbol{v_i}\}$ folded RS code
$\boldsymbol{A}$ random
$\boldsymbol{p = \exp(n \log n)}$
$\boldsymbol{|A| = p/2}$
$\boldsymbol{m \approx 6n}$

Classically hard
in the query model

DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v_i}\}$ RS code
$\boldsymbol{A}$ random
$\boldsymbol{p = 4n}$
$\boldsymbol{|A| = p/2}$
$\boldsymbol{m = 4n}$

Still seems
classically hard

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Why? We can handle worst-case $\{\boldsymbol{v}_i\}$, exponential $\boldsymbol{p}$, and $\boldsymbol{A}$ of large size
But we cannot handle $\boldsymbol{m} \ll \boldsymbol{n}^2$

Chen-Liu-Zhandry'21

Random $\{\boldsymbol{v}_i\}$
$\boldsymbol{A}$ arbitrary
$\boldsymbol{p} = \mathbf{poly}(\boldsymbol{n})$
$|\boldsymbol{A}| = \boldsymbol{p} - \boldsymbol{k} + \mathbf{1}$
$\boldsymbol{m} \approx \boldsymbol{n}^{\boldsymbol{k}}$

Now classically easy

Yamakawa-Zhandry'22

$\{\boldsymbol{v}_i\}$ folded RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{exp}(\boldsymbol{n} \log \boldsymbol{n})$
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$\boldsymbol{m} \approx \mathbf{6}\boldsymbol{n}$

Classically hard
in the query model

DQI'24, Chailloux-Tillich'24

$\{\boldsymbol{v}_i\}$ RS code
$\boldsymbol{A}$ random
$\boldsymbol{p} = \mathbf{4}\boldsymbol{n}$
$|\boldsymbol{A}| = \boldsymbol{p}/\mathbf{2}$
$\boldsymbol{m} = \mathbf{4}\boldsymbol{n}$

Still seems
classically hard

Regev-reduction quantum alg via $\boldsymbol{A}$-SIS

Why?

We can handle worst-case $\{\boldsymbol{v_i}\}$, exponential $\boldsymbol{p}$, and $\boldsymbol{A}$ of large size
But we cannot handle $\boldsymbol{m} \ll \boldsymbol{n}^2$

Chen-Liu-Zhandry'21	**Yamakawa-Zhandry'22**	**DQI'24, Chailloux-Tillich'24**
Random $\{\boldsymbol{v_i}\}$	$\{\boldsymbol{v_i}\}$ folded RS code	$\{\boldsymbol{v_i}\}$ RS code
$\boldsymbol{A}$ arbitrary	$\boldsymbol{A}$ random	$\boldsymbol{A}$ random
$\boldsymbol{p = \mathrm{poly}(n)}$	$\boldsymbol{p = \exp(n \log n)}$	$\boldsymbol{p = 4n}$
$\boldsymbol{\lvert A \rvert = p - k + 1}$	$\boldsymbol{\lvert A \rvert = p/2}$	$\boldsymbol{\lvert A \rvert = p/2}$
$\boldsymbol{m \approx n^k}$	$\boldsymbol{m \approx 6n}$	$\boldsymbol{m = 4n}$
Now classically easy	Classically hard in the query model	Still seems classically hard

Outline

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Main problem: the SIS^∞ problem

Cryptographic motivation

Full generalization: the $\boldsymbol{A}$-SIS problem

Quantum motivation

Algorithm overview

Algorithm overview

$\mathbf{F}_3^n$-Subset-Sum

Reducible vector

The SIS^∞ problem

Weight reduction

The $\boldsymbol{A}$-SIS problem

General reduction

$\mathbb{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_3^n$ where $\boldsymbol{m \approx n^2/3}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

$\mathbf{F}_3^n$-Subset-Sum

$\boldsymbol{n^2} \;\rightarrow\; \boldsymbol{n^2/2} \;\rightarrow \boldsymbol{n^2/3}$

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_3^n$ where $\boldsymbol{m \approx n^2/3}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

$v_1^{(1)}, \dots, v_{n+1}^{(1)}$	$v_1^{(2)}, \dots, v_{n+1}^{(2)}$	$\dots\dots$	$v_1^{(n+1)}, \dots, v_{n+1}^{(n+1)}$

$\mathbf{F}_3^n$-Subset-Sum

Input: $v_1, \ldots, v_m \in \mathbf{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute linear dependence in each batch

$\mathbf{F}_3^n$-Subset-Sum

Input: $v_1, \ldots, v_m \in \mathbf{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute linear dependence in each batch

$\sum_i \alpha_i v_i^{(1)} = \vec{0}$ where $\alpha_i \in \{0, 1, -1\}$ not all-0

$v_1^{(1)}, \ldots, v_{n+1}^{(1)}$

$\mathbf{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$ where $m = (n+1)^2 \approx n^2$

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Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute linear dependence in each batch

$\sum_i \alpha_i v_i^{(1)} = \vec{0}$ where $\alpha_i \in \{0, 1, -1\}$ not all-$\mathbf{0}$

$v_1^{(1)}, \dots, v_{n+1}^{(1)}$

Define $u^{(1)} = \sum_{i:\alpha_i=1} v_i^{(1)}$ and $w^{(1)} = \sum_{i:\alpha_i=-1} v_i^{(1)}$

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute linear dependence in each batch

$v_1^{(1)}, \dots, v_{n+1}^{(1)}$

$\sum_i \alpha_i v_i^{(1)} = \vec{0}$ where $\alpha_i \in \{0, 1, -1\}$ not all-$\mathbf{0}$

Define $u^{(1)} = \sum_{i:\alpha_i=1} v_i^{(1)}$ and $w^{(1)} = \sum_{i:\alpha_i=-1} v_i^{(1)}$

Then $u^{(1)} = w^{(1)}$ are disjoint subset-sum in this batch

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute **2** disjoint subset-sums that are equal in each batch

$v_1^{(1)}, \dots, v_{n+1}^{(1)}$

$\sum_i \alpha_i v_i^{(1)} = \vec{0}$ where $\alpha_i \in \{0, 1, -1\}$ not all-$\mathbf{0}$

Define $u^{(1)} = \sum_{i:\alpha_i=1} v_i^{(1)}$ and $w^{(1)} = \sum_{i:\alpha_i=-1} v_i^{(1)}$

Then $u^{(1)} = w^{(1)}$ are disjoint subset-sum in this batch

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute **2** disjoint subset-sums that are equal in each batch

$v_1^{(1)}, \dots, v_{n+1}^{(1)}$

$\sum_i \alpha_i v_i^{(1)} = \vec{0}$ where $\alpha_i \in \{0, 1, -1\}$ not all-$\mathbf{0}$

Define $u^{(1)} = \sum_{i:\alpha_i=1} v_i^{(1)}$ and $w^{(1)} = \sum_{i:\alpha_i=-1} v_i^{(1)}$

Then $u^{(1)} = w^{(1)}$ are disjoint subset-sum in this batch

If $u^{(1)} = w^{(1)} = \vec{0}$, we are done

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute **2** disjoint subset-sums that are equal in each batch

$v_1^{(1)}, \dots, v_{n+1}^{(1)}$	$v_1^{(2)}, \dots, v_{n+1}^{(2)}$	……	$v_1^{(n+1)}, \dots, v_{n+1}^{(n+1)}$
$\Downarrow$	$\Downarrow$		$\Downarrow$
$u^{(1)} = w^{(1)}$	$u^{(2)} = w^{(2)}$	……	$u^{(n+1)} = w^{(n+1)}$

$\mathbf{F}_3^n$-Subset-Sum

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute **2** disjoint subset-sums $u^{(i)} = w^{(i)}$ in each batch

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_3^n$ where $\boldsymbol{m} = (\boldsymbol{n} + \mathbf{1})^2 \approx \boldsymbol{n}^2$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Partition $\boldsymbol{m}$ input vectors into $\boldsymbol{n} + \mathbf{1}$ batches of $\boldsymbol{n} + \mathbf{1}$ vectors

Compute **2** disjoint subset-sums $\boldsymbol{u}^{(i)} = \boldsymbol{w}^{(i)}$ in each batch

Compute linear dependence of the subset-sums

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_3^n$ where $\boldsymbol{m = (n+1)^2 \approx n^2}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Partition $\boldsymbol{m}$ input vectors into $\boldsymbol{n+1}$ batches of $\boldsymbol{n+1}$ vectors

Compute **2** disjoint subset-sums $\boldsymbol{u}^{(i)} = \boldsymbol{w}^{(i)}$ in each batch

Compute linear dependence of the subset-sums

$$\boldsymbol{\beta}_1 \boldsymbol{u}^{(1)} + \boldsymbol{\beta}_2 \boldsymbol{u}^{(2)} + \cdots + \boldsymbol{\beta}_{n+1} \boldsymbol{u}^{(n+1)} = \vec{\mathbf{0}} \quad \text{where} \quad \boldsymbol{\beta}_i \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \text{ not all-}\mathbf{0}$$

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_3^n$ where $\boldsymbol{m = (n+1)^2 \approx n^2}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Partition $\boldsymbol{m}$ input vectors into $\boldsymbol{n+1}$ batches of $\boldsymbol{n+1}$ vectors

Compute **2** disjoint subset-sums $\boldsymbol{u^{(i)}} = \boldsymbol{w^{(i)}}$ in each batch

Compute linear dependence of the subset-sums

$$\boldsymbol{\beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_{n+1} u^{(n+1)}} = \vec{\mathbf{0}} \quad \text{where} \quad \boldsymbol{\beta_i} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \text{ not all-}\mathbf{0}$$

$$\boldsymbol{\beta_i} = \begin{cases} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{cases}$$

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_3^n$ where $\boldsymbol{m} = (\boldsymbol{n}+\mathbf{1})^2 \approx \boldsymbol{n}^2$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Partition $\boldsymbol{m}$ input vectors into $\boldsymbol{n}+\mathbf{1}$ batches of $\boldsymbol{n}+\mathbf{1}$ vectors

Compute **2** disjoint subset-sums $\boldsymbol{u}^{(i)} = \boldsymbol{w}^{(i)}$ in each batch

Compute linear dependence of the subset-sums

$$\boldsymbol{\beta}_1\boldsymbol{u}^{(1)} + \boldsymbol{\beta}_2\boldsymbol{u}^{(2)} + \cdots + \boldsymbol{\beta}_{n+1}\boldsymbol{u}^{(n+1)} = \vec{\mathbf{0}} \quad \text{where} \quad \boldsymbol{\beta}_i \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \text{ not all-}\mathbf{0}$$

$$\boldsymbol{\beta}_i = \begin{cases} \mathbf{0} & \boldsymbol{\beta}_i\boldsymbol{u}^{(i)} = \vec{\mathbf{0}} \text{ is a trivial subset-sum} \\ \mathbf{1} & \\ \mathbf{2} & \end{cases}$$

$\mathbb{F}_3^n$-Subset-Sum

Input: $v_1, \ldots, v_m \in \mathbb{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute **2** disjoint subset-sums $u^{(i)} = w^{(i)}$ in each batch

Compute linear dependence of the subset-sums

$$\beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_{n+1} u^{(n+1)} = \vec{0} \quad \text{where} \quad \beta_i \in \{0, 1, 2\} \text{ not all-}0$$

$$\beta_i = \begin{cases} 0 & \beta_i u^{(i)} = \vec{0} \text{ is a trivial subset-sum} \\ 1 & \beta_i u^{(i)} = u^{(i)} \text{ is a subset-sum} \\ 2 & \end{cases}$$

$\mathbb{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbb{F}_3^n$ where $\boldsymbol{m = (n+1)^2 \approx n^2}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Partition $\boldsymbol{m}$ input vectors into $\boldsymbol{n+1}$ batches of $\boldsymbol{n+1}$ vectors

Compute **2** disjoint subset-sums $\boldsymbol{u^{(i)} = w^{(i)}}$ in each batch

Compute linear dependence of the subset-sums

$$\boldsymbol{\beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_{n+1} u^{(n+1)} = \vec{0}} \quad \text{where} \quad \boldsymbol{\beta_i} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \text{ not all-}\mathbf{0}$$

$$\boldsymbol{\beta_i} = \begin{cases} \mathbf{0} & \boldsymbol{\beta_i u^{(i)}} = \vec{\mathbf{0}} \text{ is a trivial subset-sum} \\ \mathbf{1} & \boldsymbol{\beta_i u^{(i)} = u^{(i)}} \text{ is a subset-sum} \\ \mathbf{2} & \boldsymbol{\beta_i u^{(i)} = 2u^{(i)} = u^{(i)} + w^{(i)}} \text{ is a subset-sum} \end{cases}$$

$\mathbf{F}_3^n$-Subset-Sum

Input: $v_1, \ldots, v_m \in \mathbf{F}_3^n$ where $m = (n+1)^2 \approx n^2$

Output: a nontrivial subset that sums to $\vec{0}$

Partition m input vectors into $n+1$ batches of $n+1$ vectors

Compute 2 disjoint subset-sums $u^{(i)} = w^{(i)}$ in each batch

Compute linear dependence of the subset-sums

$$\beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_{n+1} u^{(n+1)} = \vec{0} \quad \text{where} \quad \beta_i \in \{0, 1, 2\} \text{ not all-}0$$

Substitute back

$$\beta_i = \begin{cases} 0 & \beta_i u^{(i)} = \vec{0} \text{ is a trivial subset-sum} \\ 1 & \beta_i u^{(i)} = u^{(i)} \text{ is a subset-sum} \\ 2 & \beta_i u^{(i)} = 2u^{(i)} = u^{(i)} + w^{(i)} \text{ is a subset-sum} \end{cases}$$

$\mathbf{F}_3^n$-Subset-Sum

Input: $v_1, \ldots, v_m \in \mathbf{F}_3^n$ where $m \approx n^2 \rightarrow n^2/2$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

$\mathbf{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

$\mathbf{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

$v_1, \dots, v_{n+1}$

$v_{n+2}, v_{n+3}, \dots, v_m$

$\mathbf{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$u = w$

$v_{n+2}, v_{n+3}, \dots, v_m$

$0 \cdot u = \vec{0}$ is a subset-sum

$1 \cdot u = u$ is a subset-sum

$2 \cdot u = u + w$ is a subset-sum

$\mathbb{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$u = w$

$0 \cdot u = \vec{0}$ is a subset-sum

$1 \cdot u = u$ is a subset-sum

$2 \cdot u = u + w$ is a subset-sum

$v_{n+2}, v_{n+3}, \dots, v_m$

$v_j = c_j u + v'_j$ where

- $v'_j \in u^{\perp}$ (complement space of u)
- $c_j = 0, 1, 2$

$\mathbb{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$u = w$

$0 \cdot u = \vec{0}$ is a subset-sum
$1 \cdot u = u$ is a subset-sum
$2 \cdot u = u + w$ is a subset-sum

$v_{n+2}, v_{n+3}, \dots, v_m$

$v_j = c_j u + v_j'$ where

- $v_j' \in u^\perp$ (complement space of u)
- $c_j = 0, 1, 2$

A subset sums to $\vec{0}$ in $u^\perp$ is a multiple of u in the whole space

$\mathbb{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$u = w$

$0 \cdot u = \vec{0}$ is a subset-sum
$1 \cdot u = u$ is a subset-sum
$2 \cdot u = u + w$ is a subset-sum

$v_{n+2}, v_{n+3}, \dots, v_m$

$v_j = c_j u + v_j'$ where

- $v_j' \in u^\perp$ (complement space of u)
- $c_j = 0, 1, 2$

A subset sums to $\vec{0}$ in $u^\perp$ is a multiple of u in the whole space

$\mathbb{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$u = w$

$0 \cdot u = \vec{0}$ is a subset-sum
$1 \cdot u = u$ is a subset-sum
$2 \cdot u = u + w$ is a subset-sum

$v_{n+2}, v_{n+3}, \dots, v_m$

$v_j = c_j u + v_j'$ where

- $v_j' \in u^\perp$ (complement space of u)
- $c_j = 0, 1, 2$

One dimension smaller

A subset sums to $\vec{0}$ in $u^\perp$ is a multiple of u in the whole space

$\mathbb{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$v_{n+2}, v_{n+3}, \dots, v_m$

$$\underbrace{n+1 \qquad\qquad \text{\#vectors needed for dim } n-1}_{\text{\#vectors needed for dim } n}$$

$\mathbb{F}_3^n$-Subset-Sum

Dimension reduction

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2$

Output: a nontrivial subset that sums to $\mathbf{0}$

$v_1, \dots, v_{n+1}$ | $v_{n+2}, v_{n+3}, \dots, v_m$

$$\underbrace{n + 1 \qquad \text{\#vectors needed for dim } n - 1}_{\text{\#vectors needed for dim } n}$$

$\mathbf{F}_3^n$-Subset-Sum

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_3^n$ where $\boldsymbol{m \approx n^2 \to n^2/2 \to n^2/3}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

$\mathbf{F}_3^n$-Subset-Sum

Explore sparsity

Input: $v_1, \dots, v_m \in \mathbf{F}_3^n$ where $m \approx n^2 \to n^2/2 \to n^2/3$

Output: a nontrivial subset that sums to $\vec{0}$

$\mathbb{F}_3^n$-Subset-Sum

Explore sparsity

Input: $v_1, \dots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2 \to n^2/3$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \dots, v_{n+1}$

$$u = w$$

$0 \cdot u = \vec{0}$ is a subset-sum
$1 \cdot u = u$ is a subset-sum
$2 \cdot u = u + w$ is a subset-sum

$\mathbb{F}_3^n$-Subset-Sum

Explore sparsity

Input: $v_1, \ldots, v_m \in \mathbb{F}_3^n$ where $m \approx n^2 \to n^2/2 \to n^2/3$

Output: a nontrivial subset that sums to $\vec{0}$

$v_1, \ldots, v_{n+1}$

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Def (reducible vector).
u is reducible in $T \subseteq [m]$ if any multiple of u is a subset-sum of vectors in T

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Claim.
Reducible vector u exists with $|T| \approx 2n/3$ whenever $m \geq n + \log n$

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Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_3^n$ where $\boldsymbol{m \approx n^2 \to n^2/2 \to n^2/3}$

Output: a nontrivial subset that sums to $\vec{\mathbf{0}}$

Claim.
Reducible vector $\boldsymbol{u}$ exists with $|\boldsymbol{T}| \approx \boldsymbol{2n/3}$ whenever $\boldsymbol{m} \geq \boldsymbol{n} + \mathbf{log}\, \boldsymbol{n}$

$\mathbf{F}_3^n$-Subset-Sum

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Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_3^n$ where $\boldsymbol{m} \approx \boldsymbol{n}^2 \to \boldsymbol{n}^2/2 \to \boldsymbol{n}^2/3$

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$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$

Reducible $\boldsymbol{u}$

Find $\vec{\mathbf{0}}$ in $\boldsymbol{u}^{\perp}$ (dim $\boldsymbol{n} - \mathbf{1}$)

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$v_1, \dots, v_m$

$v_1, \dots, v_{2n/3}$ $\Rightarrow$ Reducible u

$v_{2n/3+1}, \dots, v_m$ $\Rightarrow$ Find $\vec{0}$ in $u^\perp$ (dim $n-1$)

$\mathbf{F}_3^n$-Subset-Sum

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Claim.
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$$\frac{n^2}{3} = \frac{n^2}{2} \cdot \frac{2}{3}$$

$v_1, \dots, v_m$

$v_1, \dots, v_{2n/3}$ | $v_{2n/3+1}, \dots, v_m$

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$\sum_{i \in T} \alpha_i v_i = \vec{0}$ where $\alpha_i \in \{1, -1\}$

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$\sum_{i \in T} \alpha_i v_i = \vec{0}$ where $\alpha_i \in \{1, -1\}$

Let $u = \sum_{i:\alpha_i = 1} v_i$ and $w = \sum_{i:\alpha_i = -1} v_i$

Then $u = w$ are disjoint subset-sum in $\{v_i\}$

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$\Rightarrow$ u is reducible

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$v_1, \dots, v_n$

$y_1, \dots, y_k$ for $k = \log n$

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Basis change

$e_1, \dots, e_n$

$z_1, \dots, z_k$

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Random linear comb with $\alpha_1, \dots, \alpha_k \in \mathbb{F}_3$

In expectation $z = \sum_i \alpha_i z_i$ has at most $2n/3$ nonzero entries

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Some explicit nontrivial $z = \sum_i \alpha_i z_i$ has at most $\frac{2n/3}{1-3^{-k}} \approx 2n/3$ nonzero entries

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Linear dependence exists with $|T| \approx 2n/3$ whenever $m \geq n + \log n$

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Linear dependence using $\approx 2n/3 + k \approx 2n/3$ vectors

Open problem on $\mathbb{F}_3^n$-Subset-Sum

Given $m \approx n^2/3$ vectors in $\mathbb{F}_3^n$, we can efficiently find a nontrivial subset that sums to $\vec{\mathbf{0}}$

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Is $m \ll n^2/3$ possible?

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Given $m \approx n^2/3$ vectors in $\mathbb{F}_3^n$, we can efficiently find a nontrivial subset that sums to $\vec{0}$

Is $m \ll n^2/3$ possible?

$m = 2n + 1$ guarantees solution

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Average case?

Given $m \approx (1 + o(1))n$ vectors in $\mathbf{F}_3^n$, we can efficiently find linear dependence that uses only $R \approx 2n/3$ vectors

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Given $m \approx n^2/3$ vectors in $\mathbf{F}_3^n$, we can efficiently find a nontrivial subset that sums to $\vec{\mathbf{0}}$

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$m = 2n + 1$ guarantees solution

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Is $m \ll n^2/3$ possible?

$m = 2n + 1$ guarantees solution

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Given $m \approx \big(1 + o(1)\big)n$ vectors in $\mathbb{F}_3^n$, we can efficiently find linear dependence that uses only $R \approx 2n/3$ vectors

Is $R \ll 2n/3$ possible?

No, consider $e_1, \dots, e_n$ and $o(n)$ random vectors

Open problem on $\mathbf{F}_3^n$-Subset-Sum

Given $m \approx n^2/3$ vectors in $\mathbf{F}_3^n$, we can efficiently find a nontrivial subset that sums to $\vec{\mathbf{0}}$

Is $m \ll n^2/3$ possible?

$m = 2n + 1$ guarantees solution

Average case?

Given $m \approx n^{1.99}$ vectors in $\mathbf{F}_3^n$, we can efficiently find linear dependence that uses only $R \approx 2n/3$ vectors

Is $R \ll 2n/3$ possible?

Open problem on $\mathbf{F}_3^n$-Subset-Sum

Given $m \approx n^2/3$ vectors in $\mathbf{F}_3^n$, we can efficiently find a nontrivial subset that sums to $\vec{\mathbf{0}}$

Is $m \ll n^2/3$ possible?

$m = 2n + 1$ guarantees solution

Average case?

Given $m \approx n^{1.99}$ vectors in $\mathbf{F}_3^n$, we can efficiently find linear dependence that uses only $R \approx 2n/3$ vectors

Is $R \ll 2n/3$ possible?

$R \approx n/\log n$ is possible, ignoring efficiency

Open problem on $\mathbf{F}_3^n$-Subset-Sum

Given $m \approx n^{100}$ vectors in $\mathbf{F}_3^n$, we can efficiently find linear dependence that uses only $R \approx 2n/3$ vectors

Is $R \ll 2n/3$ possible?

$R \approx n/\log n$ is possible, ignoring efficiency

Open problem on $\mathbf{F}_3^n$-Subset-Sum

Given $m \approx n^{100}$ vectors in $\mathbf{F}_3^n$, we can efficiently find linear dependence that uses only $R \approx 2n/3$ vectors

Is $R \ll 2n/3$ possible?

$R \approx n/\log n$ is possible, ignoring efficiency

Given $m \approx n^{100}$ vectors in $\mathbf{F}_2^n$, we can efficiently find linear dependence that uses only $R \approx n/2$ vectors

Is $R \ll n/2$ possible?

$R \approx n/\log n$ is possible, ignoring efficiency

Algorithm overview

$\mathbf{F}_3^n$-Subset-Sum

Reducible vector

The SIS^∞ problem

Weight reduction

The $\boldsymbol{A}$-SIS problem

General reduction

The SIS^{∞} problem

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

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Input: $\boldsymbol{v_1}, \ldots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$

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Fact. If $\boldsymbol{h} = \boldsymbol{p}/\mathbf{2}$, then $\boldsymbol{m} = \boldsymbol{n} + \mathbf{1}$ suffices

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Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$

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Fact. If $\boldsymbol{h} = \boldsymbol{p}/\mathbf{2}$, then $\boldsymbol{m} = \boldsymbol{n} + \mathbf{1}$ suffices

Proof. View $\mathbf{F}_{\boldsymbol{p}}$ as $\{-\lfloor \boldsymbol{p}/\mathbf{2} \rfloor, \dots, \lfloor \boldsymbol{p}/\mathbf{2} \rfloor\}$

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Fact. If $\boldsymbol{h} = \boldsymbol{p}/2$, then $\boldsymbol{m} = \boldsymbol{n} + 1$ suffices

Lemma (weight halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$, then $\boldsymbol{m} = \boldsymbol{R}^2$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2$

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If $\boldsymbol{m = R}$ suffices for $\boldsymbol{h = B}$, then $\boldsymbol{m = R^2}$ suffices for $\boldsymbol{h = B/2}$

Theorem (Imran-Ivanyos'25).

Assume $\boldsymbol{k}$ is a power of two and

$$\boldsymbol{m \gg p^{k \log k} n^k}$$

Classical polytime algorithm for

$$\boldsymbol{h = \frac{p}{2k}}$$

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Proof.
By induction on $\boldsymbol{k = 2^i}$

The SIS$^\infty$ problem

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Classical polytime algorithm for

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Proof.

By induction on $\boldsymbol{k = 2^i}$

Base case is $\boldsymbol{i = 0}$ and $\boldsymbol{h = \frac{p}{2}}$

Then $\boldsymbol{m \geq n + 1}$ suffices

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Input: $v_1, \dots, v_m \in \mathbf{F}_p^n$

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Assume k is a power of two and

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$$h = \frac{p}{2k}$$

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By induction on $k = 2^i$
Base case is $i = 0$ and $h = \frac{p}{2}$
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Since $\boldsymbol{m = (n+1)^k}$ suffices for $\boldsymbol{h = \frac{p}{2k}}$

$\boldsymbol{m = (n+1)^{2k}}$ suffices for $\boldsymbol{h = \frac{p}{4k}}$

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The SIS$^\infty$ problem

Input: $\boldsymbol{v_1, \ldots, v_m \in \mathbf{F}_p^n}$

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Dimension reduction and **exploring sparsity** also apply

Fact. If $\boldsymbol{h = p/2}$, then $\boldsymbol{m = n + 1}$ suffices

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If $\boldsymbol{m = R}$ suffices for $\boldsymbol{h = B}$, then $\boldsymbol{m = R^2}$ suffices for $\boldsymbol{h = B/2}$

Theorem (Imran-Ivanyos'25).
Assume $\boldsymbol{k}$ is a power of two and

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$$\boldsymbol{h = \frac{p}{2k}}$$

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By induction on $\boldsymbol{k = 2^i}$
Base case is $\boldsymbol{i = 0}$ and $\boldsymbol{h = \frac{p}{2}}$
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Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

Compute linear dependence of $\{\boldsymbol{u}^{(i)}\}$ and substitute back

Lemma (weight halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$,
then $\boldsymbol{m} = \boldsymbol{R}^2$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2$

The SIS^∞ problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

Compute linear dependence of $\{\boldsymbol{u}^{(i)}\}$ and substitute back

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm\boldsymbol{B}/2$

Lemma (weight halving).
If $m = R$ suffices for $h = B$,
then $m = R^2$ suffices for $h = B/2$

The SIS^{∞} problem

Input: $v_1, \ldots, v_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Partition R^2 vectors into R batches of R vectors

Compute *reducible vector* $u^{(i)}$ in each batch

Compute linear dependence of $\{u^{(i)}\}$ and substitute back

Def (reducible vector).
$u^{(i)}$ is reducible if for any $-B \leq c \leq B$,
$c \cdot u^{(i)}$ is a linear comb of vectors in batch i
using coeffs in $\pm B/2$

$$\beta_1 u^{(1)} + \cdots + \beta_R u^{(R)} = \vec{0}$$
where $-B \leq \beta_i \leq B$ not all-$\mathbf{0}$

Lemma (weight halving).
If $m = R$ suffices for $h = B$,
then $m = R^2$ suffices for $h = B/2$

The SIS^∞ problem

Input: $v_1, \ldots, v_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Partition R^2 vectors into R batches of R vectors

Compute *reducible vector* $u^{(i)}$ in each batch

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Def (reducible vector).
$u^{(i)}$ is reducible if for any $-B \leq c \leq B$,
$c \cdot u^{(i)}$ is a linear comb of vectors in batch i
using coeffs in $\pm B/2$

$$\beta_1 u^{(1)} + \cdots + \beta_R u^{(R)} = \vec{0}$$

where $-B \leq \beta_i \leq B$ not all-$\mathbf{0}$

Replace each $\beta_i u^{(i)}$ by reducibility

The SIS^∞ problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm\boldsymbol{B}/\boldsymbol{2}$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

The SIS^∞ problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$ using coeffs in $\pm\boldsymbol{B}/\mathbf{2}$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$\boldsymbol{v}_1^{(i)}, \ldots, \boldsymbol{v}_R^{(i)}$ $\longrightarrow$ $\boldsymbol{c}_1\boldsymbol{v}_1^{(i)} + \cdots + \boldsymbol{c}_R\boldsymbol{v}_R^{(i)} = \vec{\mathbf{0}}$
where $-\boldsymbol{B} \leq \boldsymbol{c}_j \leq \boldsymbol{B}$ not all-$\mathbf{0}$

Since $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$

The SIS$^\infty$ problem

Input: $v_1, \dots, v_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Def (reducible vector).
$u^{(i)}$ is reducible if for any $-B \le c \le B$,
$c \cdot u^{(i)}$ is a linear comb of vectors in batch i using coeffs in $\pm B/2$

Partition R^2 vectors into R batches of R vectors

Compute *reducible vector* $u^{(i)}$ in each batch

$v_1^{(i)}, \dots, v_R^{(i)}$ $\rightarrow$ $c_1 v_1^{(i)} + \cdots + c_R v_R^{(i)} = \vec{0}$ where $-B \le c_j \le B$ not all-0

Since $m = R$ suffices for $h = B$

$\downarrow$

$1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B = \vec{0}$

where $T_s = \sum_{j:c_j=s} v_j^{(i)} + \sum_{j:c_j=-s} -v_j^{(i)}$

The SIS^{∞} problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j:c_j=s} \boldsymbol{v}_j^{(i)} + \sum_{j:c_j=-s} -\boldsymbol{v}_j^{(i)}$

The SIS^∞ problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j: c_j = s} \boldsymbol{v}_j^{(i)} + \sum_{j: c_j = -s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

The SIS$^\infty$ problem

Input: $\boldsymbol{v_1, \dots, v_m \in \mathbb{F}_p^n}$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u^{(i)}}$ is reducible if for any $\boldsymbol{-B \le c \le B}$,
$\boldsymbol{c \cdot u^{(i)}}$ is a linear comb of vectors in batch $\boldsymbol{i}$ using coeffs in $\pm \boldsymbol{B/2}$

Partition $\boldsymbol{R^2}$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u^{(i)}}$ in each batch

$$\boldsymbol{1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B = \vec{0}}$$

where $\boldsymbol{T_s = \sum_{j: c_j = s} v_j^{(i)} + \sum_{j: c_j = -s} -v_j^{(i)}}$

Define $\boldsymbol{u^{(i)} = T_{B/2} + T_{B/2+1} + \cdots + T_B}$

$$\boldsymbol{c \cdot u^{(i)}} \begin{cases} \boldsymbol{0 \le c \le B/2} \\ \\ \end{cases}$$

The SIS^{∞} problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm\boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j:c_j=s} \boldsymbol{v}_j^{(i)} + \sum_{j:c_j=-s} -\boldsymbol{v}_j^{(i)}$

$$\text{Define } \boldsymbol{u}^{(i)} = \overbrace{\boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B}^{\text{Coeffs } \pm 1}$$

$$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)} \begin{cases} \mathbf{0} \leq \boldsymbol{c} \leq \boldsymbol{B}/2 \\ \end{cases}$$

has coeffs $\pm\boldsymbol{c} \subseteq \pm\boldsymbol{B}/2$

The SIS$^\infty$ problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$ using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j:c_j=s} \boldsymbol{v}_j^{(i)} + \sum_{j:c_j=-s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

$$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)} \begin{cases} \mathbf{0} \leq \boldsymbol{c} \leq \boldsymbol{B}/\mathbf{2} & \checkmark \\ \boldsymbol{B}/\mathbf{2} < \boldsymbol{c} \leq \boldsymbol{B} & \end{cases}$$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$ using coeffs in $\pm \boldsymbol{B}/2$

The SIS$^{\infty}$ problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j:c_j=s} \boldsymbol{v}_j^{(i)} + \sum_{j:c_j=-s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

$$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)} \begin{cases} \mathbf{0} \leq \boldsymbol{c} \leq \boldsymbol{B}/2 & \checkmark \\ \boldsymbol{B}/2 < \boldsymbol{c} \leq \boldsymbol{B} \end{cases}$$

$$= \boldsymbol{c} \cdot \boldsymbol{u}^{(i)} - (\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B)$$

The SIS$^{\infty}$ problem

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B = \vec{0}$$

where $T_s = \sum_{j:c_j=s} v_j^{(i)} + \sum_{j:c_j=-s} -v_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = T_{B/2} + T_{B/2+1} + \cdots + T_B$

$$c \cdot u^{(i)} \begin{cases} 0 \leq c \leq B/2 & \checkmark \\ B/2 < c \leq B \end{cases}$$

$$= c \cdot u^{(i)} - (1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B)$$
$$= \sum_{s<B/2} (-s) \cdot T_s + \sum_{s \geq B/2} (c-s) \cdot T_s$$

The SIS^∞ problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbb{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$1 \cdot \boldsymbol{T}_1 + 2 \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_{\boldsymbol{B}} = \vec{0}$$

where $\boldsymbol{T}_s = \sum_{j: c_j = s} \boldsymbol{v}_j^{(i)} + \sum_{j: c_j = -s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

$$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)} \begin{cases} 0 \le \boldsymbol{c} \le \boldsymbol{B}/2 & ✓ \\ \boldsymbol{B}/2 < \boldsymbol{c} \le \boldsymbol{B} \end{cases}$$

$$= \boldsymbol{c} \cdot \boldsymbol{u}^{(i)} - (1 \cdot \boldsymbol{T}_1 + 2 \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B)$$

$$= \sum_{s < B/2} (-s) \cdot \boldsymbol{T}_s + \sum_{s \ge B/2} (\boldsymbol{c} - s) \cdot \boldsymbol{T}_s$$

has coeffs $\pm \boldsymbol{B}/2$

since $\boldsymbol{s} \le \boldsymbol{B}/2$ and $|\boldsymbol{c} - \boldsymbol{s}| \le \boldsymbol{B}/2$

and $\boldsymbol{T}_s$ has coeff ± 1

The SIS$^\infty$ problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j: c_j = s} \boldsymbol{v}_j^{(i)} + \sum_{j: c_j = -s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

$$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)} \begin{cases} \mathbf{0} \le \boldsymbol{c} \le \boldsymbol{B}/\mathbf{2} & \checkmark \\ \boldsymbol{B}/\mathbf{2} < \boldsymbol{c} \le \boldsymbol{B} & \checkmark \end{cases}$$

The SIS$^\infty$ problem

Input: $v_1, \dots, v_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Def (reducible vector).
$u^{(i)}$ is reducible if for any $-B \le c \le B$,
$c \cdot u^{(i)}$ is a linear comb of vectors in batch i
using coeffs in $\pm B/2$

Partition R^2 vectors into R batches of R vectors

Compute *reducible vector* $u^{(i)}$ in each batch

$$1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B = \vec{0}$$

where $T_s = \sum_{j:c_j=s} v_j^{(i)} + \sum_{j:c_j=-s} -v_j^{(i)}$

Define $u^{(i)} = T_{B/2} + T_{B/2+1} + \cdots + T_B$

$$c \cdot u^{(i)} \begin{cases} 0 \le |c| \le B/2 & \checkmark \\ B/2 < |c| \le B & \checkmark \end{cases}$$

The SIS$^\infty$ problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$ using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j: c_j = s} \boldsymbol{v}_j^{(i)} + \sum_{j: c_j = -s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

$$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)} \begin{cases} \mathbf{0} \leq |\boldsymbol{c}| \leq \boldsymbol{B}/2 & \checkmark \\ \boldsymbol{B}/2 < |\boldsymbol{c}| \leq \boldsymbol{B} & \checkmark \end{cases}$$

$\boldsymbol{u}^{(i)}$ is reducible

The SIS$^\infty$ problem

Input: $v_1, \dots, v_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Def (reducible vector).
$u^{(i)}$ is reducible if for any $-B \leq c \leq B$,
$c \cdot u^{(i)}$ is a linear comb of vectors in batch i
using coeffs in $\pm B/2$

Partition R^2 vectors into R batches of R vectors

Compute *reducible vector* $u^{(i)}$ in each batch

$$1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B = \vec{0}$$

where $T_s = \sum_{j:c_j=s} v_j^{(i)} + \sum_{j:c_j=-s} -v_j^{(i)}$

Define $u^{(i)} = T_{B/2} + T_{B/2+1} + \cdots + T_B$

What if $T_{B/2} + T_{B/2+1} + \cdots + T_B$ is an empty sum?

The SIS$^\infty$ problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Def (reducible vector).
$\boldsymbol{u}^{(i)}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}^{(i)}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B}/2$

Partition $\boldsymbol{R}^2$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R}$ vectors

Compute *reducible vector* $\boldsymbol{u}^{(i)}$ in each batch

$$1 \cdot \boldsymbol{T}_1 + 2 \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\mathbf{0}}$$

where $\boldsymbol{T}_s = \sum_{j:c_j=s} \boldsymbol{v}_j^{(i)} + \sum_{j:c_j=-s} -\boldsymbol{v}_j^{(i)}$

Define $\boldsymbol{u}^{(i)} = \boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$

What if $\boldsymbol{T}_{B/2} + \boldsymbol{T}_{B/2+1} + \cdots + \boldsymbol{T}_B$ is an empty sum?

$$\boldsymbol{c}_1 \boldsymbol{v}_1^{(i)} + \cdots + \boldsymbol{c}_R \boldsymbol{v}_R^{(i)} = \vec{\mathbf{0}}$$

where $-\boldsymbol{B}/2 \leq \boldsymbol{c}_j \leq \boldsymbol{B}/2$ not all-$\mathbf{0}$

The SIS$^\infty$ problem

Input: $v_1, \dots, v_m \in \mathbb{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Def (reducible vector).
$u^{(i)}$ is reducible if for any $-B \le c \le B$,
$c \cdot u^{(i)}$ is a linear comb of vectors in batch i
using coeffs in $\pm B/2$

Partition R^2 vectors into R batches of R vectors

Compute *reducible vector* $u^{(i)}$ in each batch

$$1 \cdot T_1 + 2 \cdot T_2 + \cdots + B \cdot T_B = \vec{0}$$

where $T_s = \sum_{j:c_j=s} v_j^{(i)} + \sum_{j:c_j=-s} -v_j^{(i)}$

Define $u^{(i)} = T_{B/2} + T_{B/2+1} + \cdots + T_B$

What if $T_{B/2} + T_{B/2+1} + \cdots + T_B$ is an empty sum?

$$c_1 v_1^{(i)} + \cdots + c_R v_R^{(i)} = \vec{0}$$

where $-B/2 \le c_j \le B/2$ not all-$\mathbf{0}$

Let $c_j \leftarrow 2c_j$ and try again

The SIS^∞ problem

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

Lemma (weight halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$, then $\boldsymbol{m} = \boldsymbol{R}^2$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2$

The SIS^{∞} problem

Input: $\boldsymbol{v_1, \dots, v_m} \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Lemma (weight halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$, then $\boldsymbol{m} = \boldsymbol{R}^2$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2$

Lemma (iterative halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$, then $\boldsymbol{m} = \boldsymbol{R}^{2^t}$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2^t$

The SIS$^\infty$ problem

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Lemma (weight halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$, then $\boldsymbol{m} = \boldsymbol{R}^2$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2$

Lemma (iterative halving).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$, then $\boldsymbol{m} = \boldsymbol{R}^{2^t}$ suffices for $\boldsymbol{h} = \boldsymbol{B}/2^t$

How about dividing $\mathbf{3}$?

Do we have to pay $\boldsymbol{m} = \boldsymbol{R}^4$ and get the stronger $\boldsymbol{h} = \boldsymbol{B}/4$?

The SIS^{∞} problem

Lemma (weight trisecting).
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$,
then $\boldsymbol{m} = \boldsymbol{R^3}$ suffices for $\boldsymbol{h} = \boldsymbol{B/3}$

Input: $\boldsymbol{v_1, \dots, v_m \in \mathbf{F}_p^n}$

Output: linear dependence using coeffs in $\pm\boldsymbol{h}$

Lemma (weight trisecting).
If $m = R$ suffices for $h = B$,
then $m = R^3$ suffices for $h = B/3$

The SIS^∞ problem

Input: $v_1, \dots, v_m \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm h$

Partition R^3 vectors into R batches of R^2 vectors

Compute *reducible vector* $u^{(i)}$ in each batch

Compute linear dependence of $\{u^{(i)}\}$ and substitute back

The SIS$^\infty$ problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$

Output: linear dependence using coeffs in $\pm \boldsymbol{h}$

Lemma (weight trisecting).
If $\boldsymbol{m = R}$ suffices for $\boldsymbol{h = B}$,
then $\boldsymbol{m = R^3}$ suffices for $\boldsymbol{h = B/3}$

Partition $\boldsymbol{R^3}$ vectors into $\boldsymbol{R}$ batches of $\boldsymbol{R^2}$ vectors

Compute *reducible vector* $\boldsymbol{u^{(i)}}$ in each batch

Compute linear dependence of $\left\{\boldsymbol{u^{(i)}}\right\}$ and substitute back

Def (reducible vector).
$\boldsymbol{u^{(i)}}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u^{(i)}}$ is a linear comb of vectors in batch $\boldsymbol{i}$
using coeffs in $\pm \boldsymbol{B/3}$

$$\boldsymbol{\beta_1 u^{(1)}} + \cdots + \boldsymbol{\beta_R u^{(R)}} = \vec{\mathbf{0}}$$

where $-\boldsymbol{B} \leq \boldsymbol{\beta_i} \leq \boldsymbol{B}$ not all-$\mathbf{0}$

Replace each $\boldsymbol{\beta_i u^{(i)}}$ by reducibility

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/\mathbf{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_{\mathbf{1}}, \ldots, \boldsymbol{v}_{R^2}$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{R}^2}$

$\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{R}}$ $\Rightarrow$ $\boldsymbol{c}_1 \boldsymbol{v}_1 + \cdots + \boldsymbol{c}_R \boldsymbol{v}_R = \vec{\boldsymbol{0}}$
where $-\boldsymbol{B} \le \boldsymbol{c}_j \le \boldsymbol{B}$ not all-$\boldsymbol{0}$

Since $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$

$\Downarrow$

$\boldsymbol{1} \cdot \boldsymbol{T}_1 + \boldsymbol{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_B = \vec{\boldsymbol{0}}$
where $\boldsymbol{T}_s = \sum_{j:c_j=s} \boldsymbol{v}_j + \sum_{j:c_j=-s} -\boldsymbol{v}_j$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\mathbf{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{R}^2}$

$\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{R}}$

$\boldsymbol{c}_1\boldsymbol{v}_1 + \cdots + \boldsymbol{c}_R\boldsymbol{v}_R = \vec{\mathbf{0}}$
where $-\boldsymbol{B} \leq \boldsymbol{c}_j \leq \boldsymbol{B}$ not all-$\mathbf{0}$

Since $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{h} = \boldsymbol{B}$

$\mathbf{1} \cdot \boldsymbol{T}_1 + \mathbf{2} \cdot \boldsymbol{T}_2 + \cdots + \boldsymbol{B} \cdot \boldsymbol{T}_{\boldsymbol{B}} = \vec{\mathbf{0}}$
where $\boldsymbol{T}_s = \sum_{j:c_j = s} \boldsymbol{v}_j + \sum_{j:c_j=-s} -\boldsymbol{v}_j$

$$\begin{aligned}\vec{\mathbf{0}} \quad &= \textstyle\sum_{s<B/3} s \cdot T_s \\ &+ \textstyle\sum_{B/3 \leq s < 2B/3} s \cdot T_s \\ &+ \textstyle\sum_{s \geq 2B/3} s \cdot T_s\end{aligned}$$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\vec{\mathbf{0}} \quad = \sum_{s<B/3} \boldsymbol{s} \cdot \boldsymbol{T}_s + \sum_{B/3 \leq s < 2B/3} \boldsymbol{s} \cdot \boldsymbol{T}_s + \sum_{s \geq 2B/3} \boldsymbol{s} \cdot \boldsymbol{T}_s$$

Each $\boldsymbol{T}_s$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1 \dots, \boldsymbol{v}_R$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{R}^2}$

$$\vec{\mathbf{0}} \quad = \sum_{s<B/3} s \cdot T_s + \sum_{B/3 \le s < 2B/3} s \cdot T_s + \sum_{s \ge 2B/3} s \cdot T_s$$

Each $\boldsymbol{T}_s$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1 \dots, \boldsymbol{v}_{\boldsymbol{R}}$

$$\boldsymbol{x} = \begin{bmatrix} \sum_{s<B/3} T_s \\ \sum_{B/3 \le s < 2B/3} T_s \\ \sum_{s \ge 2B/3} T_s \end{bmatrix}$$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\vec{0} = \sum_{s<B/3} s \cdot T_s + \sum_{B/3 \le s < 2B/3} s \cdot T_s + \sum_{s \ge 2B/3} s \cdot T_s$$

Each $\boldsymbol{T}_s$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1 \ldots, \boldsymbol{v}_R$

$$\boldsymbol{x} = \begin{bmatrix} \sum_{s<B/3} T_s \\ \sum_{B/3 \le s < 2B/3} T_s \\ \sum_{s \ge 2B/3} T_s \end{bmatrix} \begin{array}{l} \textbf{Small} \\ \textbf{Median} \\ \textbf{Large} \end{array}$$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\boldsymbol{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\vec{\mathbf{0}} = \sum_{s<B/3} s \cdot T_s + \sum_{B/3 \leq s < 2B/3} s \cdot T_s + \sum_{s \geq 2B/3} s \cdot T_s$$

Each $\boldsymbol{T}_s$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1 \ldots, \boldsymbol{v}_R$

$$\boldsymbol{x} = \begin{bmatrix} \sum_{s<B/3} T_s \\ \sum_{B/3 \leq s < 2B/3} T_s \\ \sum_{s \geq 2B/3} T_s \end{bmatrix} \begin{matrix} \text{Small} \\ \text{Median} \\ \text{Large} \end{matrix}$$

$\boldsymbol{x}$ is a vector in $\mathbf{F}_p^{3n}$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\vec{0} \quad = \sum_{s<B/3} s \cdot T_s + \sum_{B/3 \le s < 2B/3} s \cdot T_s + \sum_{s \ge 2B/3} s \cdot T_s$$

Each $\boldsymbol{T}_s$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1 \dots, \boldsymbol{v}_R$

$$x = \begin{bmatrix} \sum_{s<B/3} T_s \\ \sum_{B/3 \le s < 2B/3} T_s \\ \sum_{s \ge 2B/3} T_s \end{bmatrix} \begin{array}{l} \text{Small} \\ \text{Median} \\ \text{Large} \end{array}$$

$\boldsymbol{x}$ is a vector in $\mathbf{F}_p^{3n}$

Expand $\boldsymbol{x}$ in terms of $\pm \boldsymbol{v}_1 \dots, \pm \boldsymbol{v}_R$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\vec{0} = \sum_{s<B/3} s \cdot T_s + \sum_{B/3 \le s < 2B/3} s \cdot T_s + \sum_{s \ge 2B/3} s \cdot T_s$$

Each T_s is a disjoint signed-subset-sum of $\boldsymbol{v}_1 \ldots, \boldsymbol{v}_R$

$$\boldsymbol{x} = \begin{bmatrix} \sum_{s<B/3} T_s \\ \sum_{B/3 \le s < 2B/3} T_s \\ \sum_{s \ge 2B/3} T_s \end{bmatrix} \begin{array}{l} \text{Small} \\ \text{Median} \\ \text{Large} \end{array}$$

$\boldsymbol{x}$ is a vector in $\mathbf{F}_p^{3n}$

Expand $\boldsymbol{x}$ in terms of $\pm\boldsymbol{v}_1 \ldots, \pm\boldsymbol{v}_R$ and combine

- **Small** ones with **small** coeffs $\mathbf{0} \sim \boldsymbol{B}/3$
- **Median** ones with **median** coeffs $\boldsymbol{B}/3 \sim 2\boldsymbol{B}/3$
- **Large** ones with **large** coeffs $2\boldsymbol{B}/3 \sim \boldsymbol{B}$

We obtain $\vec{\mathbf{0}}$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_R$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\text{Small}} \\ \boldsymbol{x}_{\text{Median}} \\ \boldsymbol{x}_{\text{Large}} \end{bmatrix}$$

$\boldsymbol{x}_{\text{Small}}, \boldsymbol{x}_{\text{Median}}, \boldsymbol{x}_{\text{Large}}$ are disjoint signed-subset-sums

Expand in terms of $\pm\boldsymbol{v}_1 \ldots, \pm\boldsymbol{v}_R$ and combine

- $\boldsymbol{x}_{\text{Small}}$ ones with **small** coeffs $\mathbf{0} \sim \boldsymbol{B}/\mathbf{3}$
- $\boldsymbol{x}_{\text{Median}}$ ones with **median** coeffs $\boldsymbol{B}/\mathbf{3} \sim \mathbf{2}\boldsymbol{B}/\mathbf{3}$
- $\boldsymbol{x}_{\text{Large}}$ ones with **large** coeffs $\mathbf{2}\boldsymbol{B}/\mathbf{3} \sim \boldsymbol{B}$

We obtain $\vec{\mathbf{0}}$

The SIS^{∞} problem

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$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\begin{array}{cccc} \boldsymbol{v}_1^{(1)}, \ldots, \boldsymbol{v}_R^{(1)} & \boldsymbol{v}_1^{(2)}, \ldots, \boldsymbol{v}_R^{(2)} & \cdots\cdots & \boldsymbol{v}_1^{(R)}, \ldots, \boldsymbol{v}_R^{(R)} \\ \downarrow & \downarrow & & \downarrow \\ \boldsymbol{x}^{(1)} & \boldsymbol{x}^{(2)} & & \boldsymbol{x}^{(R)} \end{array}$$

The SIS^∞ problem

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$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\boldsymbol{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{v}_1^{(1)}, \dots, \boldsymbol{v}_R^{(1)} \qquad \boldsymbol{v}_1^{(2)}, \dots, \boldsymbol{v}_R^{(2)} \qquad \cdots\cdots \qquad \boldsymbol{v}_1^{(R)}, \dots, \boldsymbol{v}_R^{(R)}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\underbrace{\boldsymbol{x}^{(1)} \qquad \boldsymbol{x}^{(2)} \qquad \boldsymbol{x}^{(R)}}$$

$$\boldsymbol{c}_1 \boldsymbol{x}^{(1)} + \cdots + \boldsymbol{c}_R \boldsymbol{x}^{(R)} = \vec{\boldsymbol{0}}$$

where $-\boldsymbol{B} \le \boldsymbol{c}_j \le \boldsymbol{B}$ not all-$\boldsymbol{0}$

Linear dependence of
$\boldsymbol{R}$ vectors using coeffs $\pm \boldsymbol{B}$

The SIS^{∞} problem

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Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{R}^2}$

$$\boldsymbol{c}_1 \boldsymbol{x}^{(1)} + \cdots + \boldsymbol{c}_R \boldsymbol{x}^{(R)} = \vec{\mathbf{0}}$$

where $-\boldsymbol{B} \leq \boldsymbol{c}_j \leq \boldsymbol{B}$ not all-$\mathbf{0}$

The SIS^{∞} problem

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$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/\boldsymbol{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_{\boldsymbol{1}}, \dots, \boldsymbol{v}_{\boldsymbol{R}^{\boldsymbol{2}}}$

$$\mathbf{1} \cdot \boldsymbol{x}^{(\mathbf{1})} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(\mathbf{2})} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(\mathbf{4})} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(\mathbf{7})} = \vec{\mathbf{0}}$$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\underbrace{\mathbf{1} \cdot \boldsymbol{x}^{(1)} - (\boldsymbol{B}/4) \cdot \boldsymbol{x}^{(2)}}_{\substack{\textbf{Small}\text{ coeffs} \\ \mathbf{0} \sim \boldsymbol{B}/3}} + \underbrace{(\boldsymbol{B}/2) \cdot \boldsymbol{x}^{(4)}}_{\substack{\textbf{Median}\text{ coeffs} \\ \boldsymbol{B}/3 \sim 2\boldsymbol{B}/3}} + \underbrace{(5\boldsymbol{B}/6) \cdot \boldsymbol{x}^{(7)}}_{\substack{\textbf{Large}\text{ coeffs} \\ 2\boldsymbol{B}/3 \sim \boldsymbol{B}}} = \vec{\mathbf{0}}$$

The SIS^{∞} problem

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Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\underbrace{\mathbf{1} \cdot \boldsymbol{x}^{(1)} - (\boldsymbol{B}/4) \cdot \boldsymbol{x}^{(2)}}_{\substack{\text{Small coeffs} \\ 0 \sim B/3}} + \underbrace{(\boldsymbol{B}/2) \cdot \boldsymbol{x}^{(4)}}_{\substack{\text{Median coeffs} \\ B/3 \sim 2B/3}} + \underbrace{(5\boldsymbol{B}/6) \cdot \boldsymbol{x}^{(7)}}_{\substack{\text{Large coeffs} \\ 2B/3 \sim B}} = \vec{\mathbf{0}}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\text{Small}} \\ \boldsymbol{x}_{\text{Median}} \\ \boldsymbol{x}_{\text{Large}} \end{bmatrix}$$

The SIS^{∞} problem

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Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\underbrace{\mathbf{1} \cdot \boldsymbol{x}^{(1)} - (\boldsymbol{B}/4) \cdot \boldsymbol{x}^{(2)}}_{\substack{\textbf{Small coeffs} \\ 0 \sim B/3}} + \underbrace{(\boldsymbol{B}/2) \cdot \boldsymbol{x}^{(4)}}_{\substack{\textbf{Median coeffs} \\ B/3 \sim 2B/3}} + \underbrace{(5\boldsymbol{B}/6) \cdot \boldsymbol{x}^{(7)}}_{\substack{\textbf{Large coeffs} \\ 2B/3 \sim B}} = \vec{\mathbf{0}}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\mathrm{Small}} \\ \boldsymbol{x}_{\mathrm{Median}} \\ \boldsymbol{x}_{\mathrm{Large}} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathrm{Small}} - \boldsymbol{x}^{(2)}_{\mathrm{Small}} & \boldsymbol{x}^{(4)}_{\mathrm{Small}} & \boldsymbol{x}^{(7)}_{\mathrm{Small}} \\ \boldsymbol{x}^{(1)}_{\mathrm{Median}} - \boldsymbol{x}^{(2)}_{\mathrm{Median}} & \boldsymbol{x}^{(4)}_{\mathrm{Median}} & \boldsymbol{x}^{(7)}_{\mathrm{Median}} \\ \boldsymbol{x}^{(1)}_{\mathrm{Large}} - \boldsymbol{x}^{(2)}_{\mathrm{Large}} & \boldsymbol{x}^{(4)}_{\mathrm{Large}} & \boldsymbol{x}^{(7)}_{\mathrm{Large}} \end{bmatrix}$$

The SIS^∞ problem

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$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\boldsymbol{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

The SIS^{∞} problem

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$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\mathbf{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_{\textbf{Small}}^{(1)} - \boldsymbol{x}_{\textbf{Small}}^{(2)} & \boldsymbol{x}_{\textbf{Small}}^{(4)} & \boldsymbol{x}_{\textbf{Small}}^{(7)} \\ \boldsymbol{x}_{\textbf{Median}}^{(1)} - \boldsymbol{x}_{\textbf{Median}}^{(2)} & \boldsymbol{x}_{\textbf{Median}}^{(4)} & \boldsymbol{x}_{\textbf{Median}}^{(7)} \\ \boldsymbol{x}_{\textbf{Large}}^{(1)} - \boldsymbol{x}_{\textbf{Large}}^{(2)} & \boldsymbol{x}_{\textbf{Large}}^{(4)} & \boldsymbol{x}_{\textbf{Large}}^{(7)} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\boldsymbol{0}}$ with proper **small**, **median**, **large** weights

The SIS$^\infty$ problem

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$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
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using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\textbf{Small}} - \boldsymbol{x}^{(2)}_{\textbf{Small}} & \boldsymbol{x}^{(4)}_{\textbf{Small}} & \boldsymbol{x}^{(7)}_{\textbf{Small}} \\ \boldsymbol{x}^{(1)}_{\textbf{Median}} - \boldsymbol{x}^{(2)}_{\textbf{Median}} & \boldsymbol{x}^{(4)}_{\textbf{Median}} & \boldsymbol{x}^{(7)}_{\textbf{Median}} \\ \boldsymbol{x}^{(1)}_{\textbf{Large}} - \boldsymbol{x}^{(2)}_{\textbf{Large}} & \boldsymbol{x}^{(4)}_{\textbf{Large}} & \boldsymbol{x}^{(7)}_{\textbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\boldsymbol{0}}$ with proper **small**, **median**, **large** weights

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)} = \vec{\boldsymbol{0}}$$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)} = \vec{\mathbf{0}}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\mathbf{Small}} \\ \boldsymbol{x}_{\mathbf{Median}} \\ \boldsymbol{x}_{\mathbf{Large}} \end{bmatrix}$$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{t}} \\ \boldsymbol{d} \quad \boldsymbol{e} \quad \boldsymbol{f} \\ \boldsymbol{g} \quad \boldsymbol{h} \quad \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)} = \vec{\mathbf{0}}$$

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)}_{\mathbf{Small}} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)}_{\mathbf{Small}} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)}_{\mathbf{Small}} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)}_{\mathbf{Small}} = \vec{\mathbf{0}}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\mathbf{Small}} \\ \boldsymbol{x}_{\mathbf{Median}} \\ \boldsymbol{x}_{\mathbf{Large}} \end{bmatrix}$$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{t}} \\ \boldsymbol{d} \quad \boldsymbol{e} \quad \boldsymbol{f} \\ \boldsymbol{g} \quad \boldsymbol{h} \quad \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)}_{\mathbf{Small}} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)}_{\mathbf{Small}} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)}_{\mathbf{Small}} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)}_{\mathbf{Small}} = \vec{\mathbf{0}}$$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)}_{\mathbf{Small}} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)}_{\mathbf{Small}} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)}_{\mathbf{Small}} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)}_{\mathbf{Small}} = \vec{\mathbf{0}}$$

$\boldsymbol{a}$ $\quad\quad$ $\boldsymbol{b}$ $\quad\quad$ $\boldsymbol{t}$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$1 \cdot \boldsymbol{x}^{(1)}_{\mathbf{Small}} - (\boldsymbol{B}/4) \cdot \boldsymbol{x}^{(2)}_{\mathbf{Small}} + (\boldsymbol{B}/2) \cdot \boldsymbol{x}^{(4)}_{\mathbf{Small}} + (5\boldsymbol{B}/6) \cdot \boldsymbol{x}^{(7)}_{\mathbf{Small}} = \vec{\mathbf{0}}$$

$$\mathbf{small} \cdot \boldsymbol{a} \quad \mathbf{median} \cdot \boldsymbol{b} \quad \mathbf{large} \cdot \boldsymbol{t}$$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$\mathbf{1} \cdot \boldsymbol{x}^{(1)}_{\mathbf{Small}} - (\boldsymbol{B}/\mathbf{4}) \cdot \boldsymbol{x}^{(2)}_{\mathbf{Small}} + (\boldsymbol{B}/\mathbf{2}) \cdot \boldsymbol{x}^{(4)}_{\mathbf{Small}} + (\mathbf{5}\boldsymbol{B}/\mathbf{6}) \cdot \boldsymbol{x}^{(7)}_{\mathbf{Small}} = \vec{\mathbf{0}}$$

$$\mathbf{small} \cdot \boldsymbol{a} + \mathbf{median} \cdot \boldsymbol{b} + \mathbf{large} \cdot \boldsymbol{t} = \vec{\mathbf{0}}$$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *column* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\begin{matrix} a \\ d \\ g \end{matrix}} & \begin{matrix} b \\ e \\ h \end{matrix} & \begin{matrix} t \\ f \\ i \end{matrix} \end{bmatrix} = \begin{bmatrix} x^{(1)}_{\mathrm{Small}} - x^{(2)}_{\mathrm{Small}} & x^{(4)}_{\mathrm{Small}} & x^{(7)}_{\mathrm{Small}} \\ x^{(1)}_{\mathrm{Median}} - x^{(2)}_{\mathrm{Median}} & x^{(4)}_{\mathrm{Median}} & x^{(7)}_{\mathrm{Median}} \\ x^{(1)}_{\mathrm{Large}} - x^{(2)}_{\mathrm{Large}} & x^{(4)}_{\mathrm{Large}} & x^{(7)}_{\mathrm{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *column* of $\boldsymbol{M}$ sums to $\vec{\boldsymbol{0}}$ with proper **small**, **median**, **large** weights

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\begin{matrix} \boldsymbol{a} \\ \boldsymbol{d} \\ \boldsymbol{g} \end{matrix}} & \begin{matrix} \boldsymbol{b} \\ \boldsymbol{e} \\ \boldsymbol{h} \end{matrix} & \begin{matrix} \boldsymbol{t} \\ \boldsymbol{f} \\ \boldsymbol{i} \end{matrix} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\text{Small}} - \boldsymbol{x}^{(2)}_{\text{Small}} & \boldsymbol{x}^{(4)}_{\text{Small}} & \boldsymbol{x}^{(7)}_{\text{Small}} \\ \boldsymbol{x}^{(1)}_{\text{Median}} - \boldsymbol{x}^{(2)}_{\text{Median}} & \boldsymbol{x}^{(4)}_{\text{Median}} & \boldsymbol{x}^{(7)}_{\text{Median}} \\ \boldsymbol{x}^{(1)}_{\text{Large}} - \boldsymbol{x}^{(2)}_{\text{Large}} & \boldsymbol{x}^{(4)}_{\text{Large}} & \boldsymbol{x}^{(7)}_{\text{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *column* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\text{Small}} \\ \boldsymbol{x}^{(1)}_{\text{Median}} \\ \boldsymbol{x}^{(1)}_{\text{Large}} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{x}^{(2)}_{\text{Small}} \\ \boldsymbol{x}^{(2)}_{\text{Median}} \\ \boldsymbol{x}^{(2)}_{\text{Large}} \end{bmatrix} = \boldsymbol{x}^{(2)}$$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\text{Small}} - \boldsymbol{x}^{(2)}_{\text{Small}} & \boldsymbol{x}^{(4)}_{\text{Small}} & \boldsymbol{x}^{(7)}_{\text{Small}} \\ \boldsymbol{x}^{(1)}_{\text{Median}} - \boldsymbol{x}^{(2)}_{\text{Median}} & \boldsymbol{x}^{(4)}_{\text{Median}} & \boldsymbol{x}^{(7)}_{\text{Median}} \\ \boldsymbol{x}^{(1)}_{\text{Large}} - \boldsymbol{x}^{(2)}_{\text{Large}} & \boldsymbol{x}^{(4)}_{\text{Large}} & \boldsymbol{x}^{(7)}_{\text{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *column* of $\boldsymbol{M}$ sums to $\vec{\boldsymbol{0}}$ with proper **small**, **median**, **large** weights

$$x^{(1)} = \begin{bmatrix} x^{(1)}_{\text{Small}} \\ x^{(1)}_{\text{Median}} \\ x^{(1)}_{\text{Large}} \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{d} \\ \boldsymbol{g} \end{bmatrix} \qquad \begin{bmatrix} x^{(2)}_{\text{Small}} \\ x^{(2)}_{\text{Median}} \\ x^{(2)}_{\text{Large}} \end{bmatrix} = x^{(2)}$$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boxed{\begin{matrix} a \\ d \\ g \end{matrix}} & \begin{matrix} b \\ e \\ h \end{matrix} & \begin{matrix} t \\ f \\ i \end{matrix} \end{bmatrix} = \begin{bmatrix} x^{(1)}_{\text{Small}} - x^{(2)}_{\text{Small}} & x^{(4)}_{\text{Small}} & x^{(7)}_{\text{Small}} \\ x^{(1)}_{\text{Median}} - x^{(2)}_{\text{Median}} & x^{(4)}_{\text{Median}} & x^{(7)}_{\text{Median}} \\ x^{(1)}_{\text{Large}} - x^{(2)}_{\text{Large}} & x^{(4)}_{\text{Large}} & x^{(7)}_{\text{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *column* of $\boldsymbol{M}$ sums to $\vec{\boldsymbol{0}}$ with proper **small**, **median**, **large** weights

$$x^{(1)} = \begin{bmatrix} x^{(1)}_{\text{Small}} \\ x^{(1)}_{\text{Median}} \\ x^{(1)}_{\text{Large}} \end{bmatrix} \qquad \begin{bmatrix} a \\ d \\ g \end{bmatrix} \qquad \begin{bmatrix} x^{(2)}_{\text{Small}} \\ x^{(2)}_{\text{Median}} \\ x^{(2)}_{\text{Large}} \end{bmatrix} = x^{(2)}$$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{(1)}_{\mathbf{Small}} - \boldsymbol{x}^{(2)}_{\mathbf{Small}} & \boldsymbol{x}^{(4)}_{\mathbf{Small}} & \boldsymbol{x}^{(7)}_{\mathbf{Small}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Median}} - \boldsymbol{x}^{(2)}_{\mathbf{Median}} & \boldsymbol{x}^{(4)}_{\mathbf{Median}} & \boldsymbol{x}^{(7)}_{\mathbf{Median}} \\ \boldsymbol{x}^{(1)}_{\mathbf{Large}} - \boldsymbol{x}^{(2)}_{\mathbf{Large}} & \boldsymbol{x}^{(4)}_{\mathbf{Large}} & \boldsymbol{x}^{(7)}_{\mathbf{Large}} \end{bmatrix}$$

Observation.
Every entry of $\boldsymbol{M}$ is a disjoint signed-subset-sum of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$
Every *row* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights
Every *column* of $\boldsymbol{M}$ sums to $\vec{\mathbf{0}}$ with proper **small**, **median**, **large** weights

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix}$$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{array}{ccc} \textbf{small} & \textbf{median} & \textbf{large} \end{array}$$

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{\mathbf{0}}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{array}{r c} \text{small}' & \multirow{3}{*}{$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix}$} \\ \text{median}' & \\ \text{large}' & \end{array}$$

$$= \vec{\boldsymbol{0}}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\boldsymbol{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{array}{c} \\ \text{small}' \\ \text{median}' \\ \text{large}' \end{array} \begin{array}{c} \begin{array}{ccc} \text{small} & \text{median} & \text{large} \end{array} \\ \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} \end{array} = \vec{\mathbf{0}}$$

$$= \vec{\mathbf{0}}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\begin{array}{c c} & \begin{array}{ccc} \textbf{small} & \textbf{median} & \textbf{large} \end{array} \\ \begin{array}{c} \textbf{small}' \\ \textbf{median}' \\ \textbf{large}' \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{\mathbf{0}} \\ & = \vec{\mathbf{0}} \end{array}$$

$$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

The SIS^∞ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\mathbf{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{array}{c c} & \begin{array}{ccc} \text{small} & \text{median} & \text{large} \end{array} \\ \begin{array}{c} \text{small}' \\ \text{median}' \\ \text{large}' \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \end{array}$$

$$= \vec{0}$$

$$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$$

$\mathbf{0} \le \boldsymbol{c} \le \boldsymbol{B}/\mathbf{3}$ (**small** $\boldsymbol{c}$)

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\boldsymbol{3}$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{array}{cc} & \begin{array}{ccc} \text{small} & \text{median} & \text{large} \end{array} \\ \begin{array}{c} \text{small}' \\ \text{median}' \\ \text{large}' \end{array} & \begin{bmatrix} a & b & t \\ d & e & f \\ g & h & i \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

$$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$$

$\boldsymbol{0} \le \boldsymbol{c} \le \boldsymbol{B}/\boldsymbol{3}$ (**small** $\boldsymbol{c}$)
$\boldsymbol{c} \cdot \boldsymbol{u}$ has coeffs in $\pm \boldsymbol{c} \subseteq \pm \boldsymbol{B}/\boldsymbol{3}$

The SIS$^\infty$ problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{R^2}$

$$\begin{array}{cc} & \begin{array}{ccc} \text{small} & \text{median} & \text{large} \end{array} \\ \begin{array}{c} \text{small}' \\ \text{median}' \\ \text{large}' \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

$$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$$

$\boldsymbol{0} \le \boldsymbol{c} \le \boldsymbol{B}/3$ **(small $\boldsymbol{c}$)** ✓

$\boldsymbol{B}/3 < \boldsymbol{c} \le 2\boldsymbol{B}/3$ **(median $\boldsymbol{c}$)**

$$\boldsymbol{c} \cdot \boldsymbol{u} = \boldsymbol{c} \cdot \boldsymbol{f} - \boldsymbol{c} \cdot \boldsymbol{h}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

The SIS^{∞} problem

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

Construct reducible vector $\boldsymbol{u}$ in $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{R^2}$

$$\begin{array}{cc} & \begin{array}{ccc} \textbf{small} & \textbf{median} & \textbf{large} \end{array} \\ \begin{array}{c} \textbf{small}' \\ \textbf{median}' \\ \textbf{large}' \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{\mathbf{0}} \\ & = \vec{\mathbf{0}} \end{array}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
Large $2B/3 \sim B$

$$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$$

$\mathbf{0} \le \boldsymbol{c} \le \boldsymbol{B}/\mathbf{3}$ (**small** $\boldsymbol{c}$) ✓

$\boldsymbol{B}/\mathbf{3} < \boldsymbol{c} \le \mathbf{2}\boldsymbol{B}/\mathbf{3}$ (**median** $\boldsymbol{c}$)

$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= \boldsymbol{c} \cdot \boldsymbol{f} - (\textbf{small}' \cdot \boldsymbol{t} + \textbf{median}' \cdot \boldsymbol{f} + \textbf{large}' \cdot \boldsymbol{i}) \\ &\quad -\boldsymbol{c} \cdot \boldsymbol{h} \end{aligned}$$

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$$= \vec{\mathbf{0}}$$

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$$\begin{array}{c|ccc} & \text{small} & \text{median} & \text{large} \\ \hline \text{small}' & & & \\ \text{median}' & & & \\ \text{large}' & & & \end{array} \quad \begin{bmatrix} a & b & t \\ d & e & f \\ g & h & i \end{bmatrix} = \vec{0}$$

$$= \vec{0}$$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$

$0 \le c \le B/3$ (small c) ✓

$B/3 < c \le 2B/3$ (median c)

$$\begin{aligned} & c \cdot u \\ &= c \cdot f - (\text{small}' \cdot t + \text{median}' \cdot f + \text{large}' \cdot i) \\ &\quad - c \cdot h + (\text{small} \cdot g + \text{median} \cdot h + \text{large} \cdot i) \\ &= \qquad \cdot t + \qquad \cdot g \\ &\quad + \qquad \cdot f + \qquad \cdot h \\ &\quad + \qquad \cdot i \end{aligned}$$

Small $0 \sim B/3$
Median $B/3 \sim 2B/3$
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$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= \boldsymbol{c} \cdot \boldsymbol{f} - (\mathbf{small}' \cdot \boldsymbol{t} + \mathbf{median}' \cdot \boldsymbol{f} + \mathbf{large}' \cdot \boldsymbol{i}) \\ &\quad -\boldsymbol{c} \cdot \boldsymbol{h} + (\mathbf{small} \cdot \boldsymbol{g} + \mathbf{median} \cdot \boldsymbol{h} + \mathbf{large} \cdot \boldsymbol{i}) \\ &= -\mathbf{small}' \cdot \boldsymbol{t} + \mathbf{small} \cdot \boldsymbol{g} \\ &\quad +(\boldsymbol{c} - \mathbf{median}') \cdot \boldsymbol{f} + (\mathbf{median} - \boldsymbol{c}) \cdot \boldsymbol{h} \\ &\quad +(\mathbf{large} - \mathbf{large}') \cdot \boldsymbol{i} \end{aligned}$$

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$$= \vec{0}$$

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$$\begin{aligned} & c \cdot u \\ & = c \cdot f - (\text{small}' \cdot t + \text{median}' \cdot f + \text{large}' \cdot i) \\ & \quad -c \cdot h + (\text{small} \cdot g + \text{median} \cdot h + \text{large} \cdot i) \\ & = \left.\begin{aligned} & -\text{small}' \cdot t + \text{small} \cdot g \\ & +(c - \text{median}') \cdot f + (\text{median} - c) \cdot h \\ & +(\text{large} - \text{large}') \cdot i \end{aligned}\right\} \text{Coeffs in } \pm B/3 \end{aligned}$$

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$\mathbf{2}\boldsymbol{B}/\mathbf{3} < \boldsymbol{c} \le \boldsymbol{B}$ (**Large** $\boldsymbol{c}$)

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$$\begin{array}{c|ccc} & \textbf{small} & \textbf{median} & \textbf{large} \\ \hline \textbf{small}' & & & \\ \textbf{median}' & & \left[\begin{matrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{matrix}\right] & \\ \textbf{large}' & & & \end{array} = \vec{0}$$

$$= \vec{0}$$

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$\boldsymbol{0 \le c \le B/3}$ **(small $\boldsymbol{c}$)** ✓

$\boldsymbol{B/3 < c \le 2B/3}$ **(median $\boldsymbol{c}$)** ✓

$\boldsymbol{2B/3 < c \le B}$ **(Large $\boldsymbol{c}$)**

$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= \boldsymbol{c} \cdot \boldsymbol{f} - (\mathbf{small} \cdot \boldsymbol{d} + \mathbf{median} \cdot \boldsymbol{e} + \mathbf{large} \cdot \boldsymbol{f}) \\ &\quad - \boldsymbol{c} \cdot \boldsymbol{h} + (\mathbf{small}' \cdot \boldsymbol{b} + \mathbf{median}' \cdot \boldsymbol{e} + \mathbf{large}' \cdot \boldsymbol{h}) \\ &= \left.\begin{aligned} & -\mathbf{small} \cdot \boldsymbol{d} + \mathbf{small}' \cdot \boldsymbol{b} \\ & +(\mathbf{median}' - \mathbf{median}) \cdot \boldsymbol{e} \\ & +(\boldsymbol{c} - \mathbf{large}) \cdot \boldsymbol{f} + (\mathbf{large}' - \boldsymbol{c}) \cdot \boldsymbol{h} \end{aligned}\right\} \text{Coeffs in } \pm \boldsymbol{B}/3 \end{aligned}$$

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$\boldsymbol{u}$ is reducible

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$$= \vec{0}$$

What if $\boldsymbol{f}$ and $\boldsymbol{h}$ are empty?

$$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$$

$0 \le |c| \le B/3$ (small c) ✓

$B/3 < |c| \le 2B/3$ (median c) ✓

$2B/3 < |c| \le B$ (Large c) ✓

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$\mathbf{2}\boldsymbol{B}/\mathbf{3} < |\boldsymbol{c}| \leq \boldsymbol{B}$ (**Large** $\boldsymbol{c}$) ✓

$\boldsymbol{u}$ is reducible

Need to change the base algorithm slightly to ensure nonemptiness

The SIS^{∞} problem

Linear dependence of
$\boldsymbol{R}$ vectors using coeffs $\pm\boldsymbol{B}$

Def (reducible vector).
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using coeffs in $\pm\boldsymbol{B}/\boldsymbol{k}$

$$\begin{array}{l} \textbf{small} \\ \textbf{large} \end{array} \begin{bmatrix} \blacksquare \\ \blacksquare \end{bmatrix}$$

$\boldsymbol{k} = \mathbf{2}$
$\boldsymbol{R}$ vectors

The SIS$^\infty$ problem

Linear dependence of
R vectors using coeffs $\pm B$

Def (reducible vector).
u is reducible if for any $-B \leq c \leq B$,
$c \cdot u$ is a linear comb of the given vectors
using coeffs in $\pm B/k$

small
large

$k = 2$
R vectors

small
median
large

$k = 3$

The SIS$^{\infty}$ problem

Linear dependence of
$\boldsymbol{R}$ vectors using coeffs $\pm\boldsymbol{B}$

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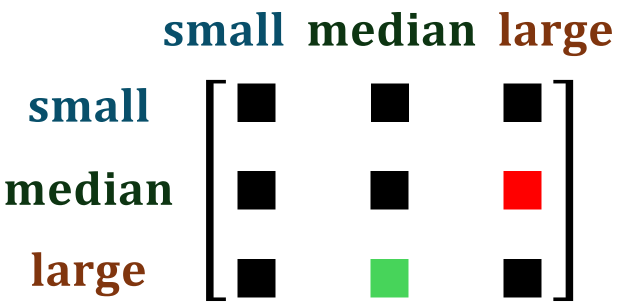
$\boldsymbol{k = 2}$
$\boldsymbol{R}$ vectors



$\boldsymbol{k = 3}$
$\boldsymbol{R^2}$ vectors

The SIS$^\infty$ problem

Linear dependence of
R vectors using coeffs $\pm B$

Def (reducible vector).
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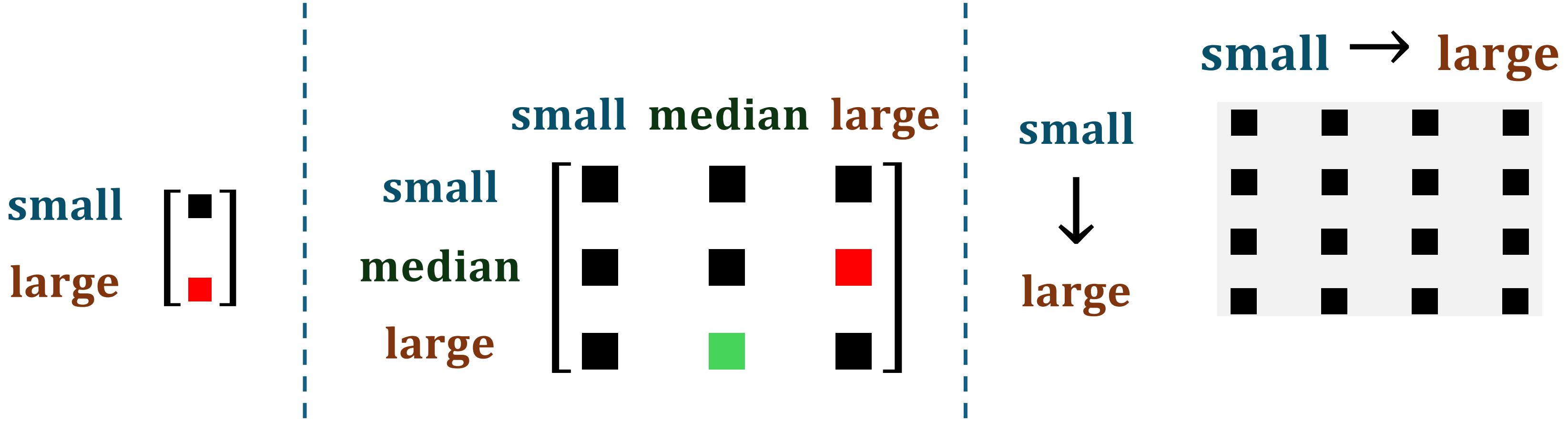

$k = 2$
R vectors

$k = 3$
R^2 vectors

$k = 4$

The SIS$^\infty$ problem

Linear dependence of
$\boldsymbol{R}$ vectors using coeffs $\pm\boldsymbol{B}$

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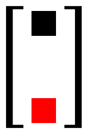
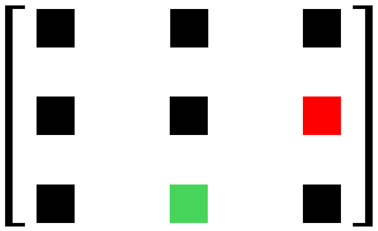
small
large



$\boldsymbol{k} = \boldsymbol{2}$
$\boldsymbol{R}$ vectors

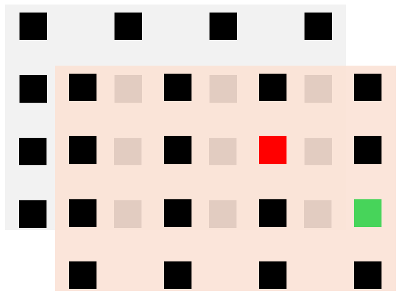
small median large
small
median
large



$\boldsymbol{k} = \boldsymbol{3}$
$\boldsymbol{R}^2$ vectors

small $\rightarrow$ large

small $\downarrow$ large

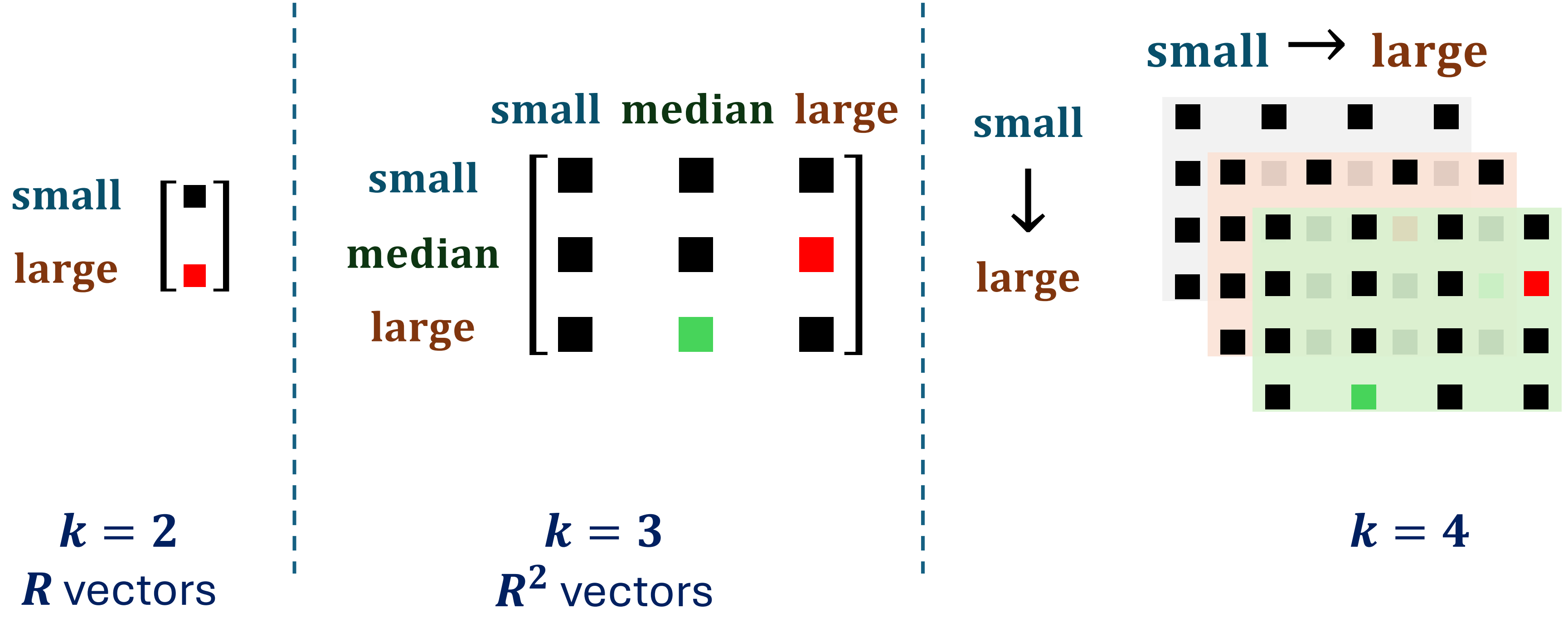


$\boldsymbol{k} = \boldsymbol{4}$

The SIS$^\infty$ problem

Linear dependence of
R vectors using coeffs $\pm B$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-B \le c \le B$,
$c \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm B/k$



The SIS$^\infty$ problem

Linear dependence of
R vectors using coeffs $\pm B$

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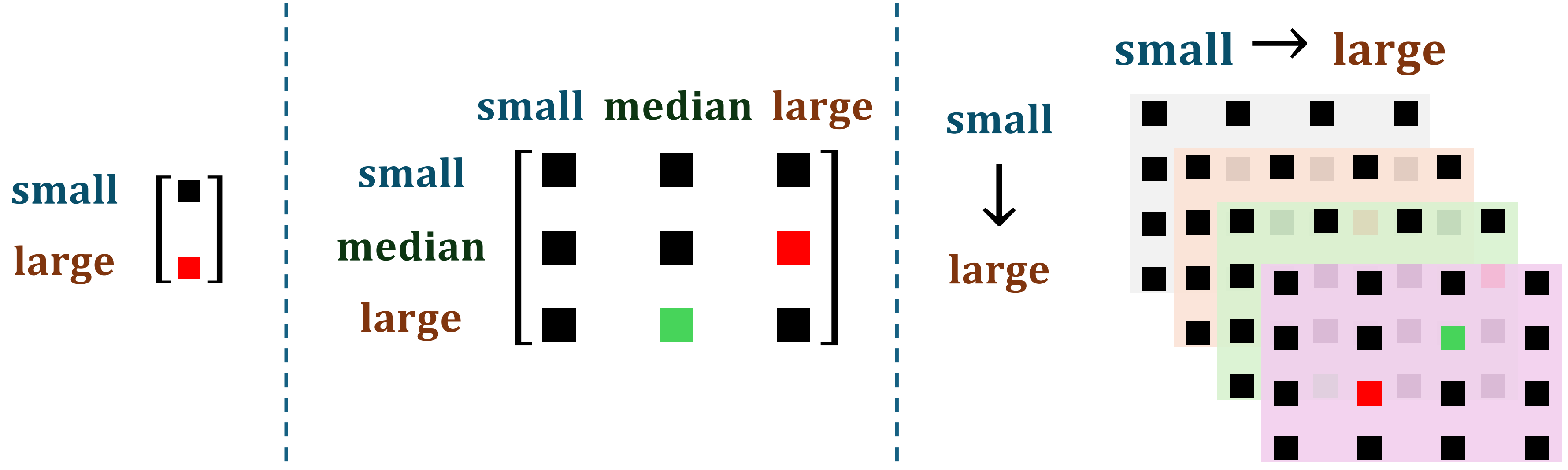


$k = 2$
R vectors

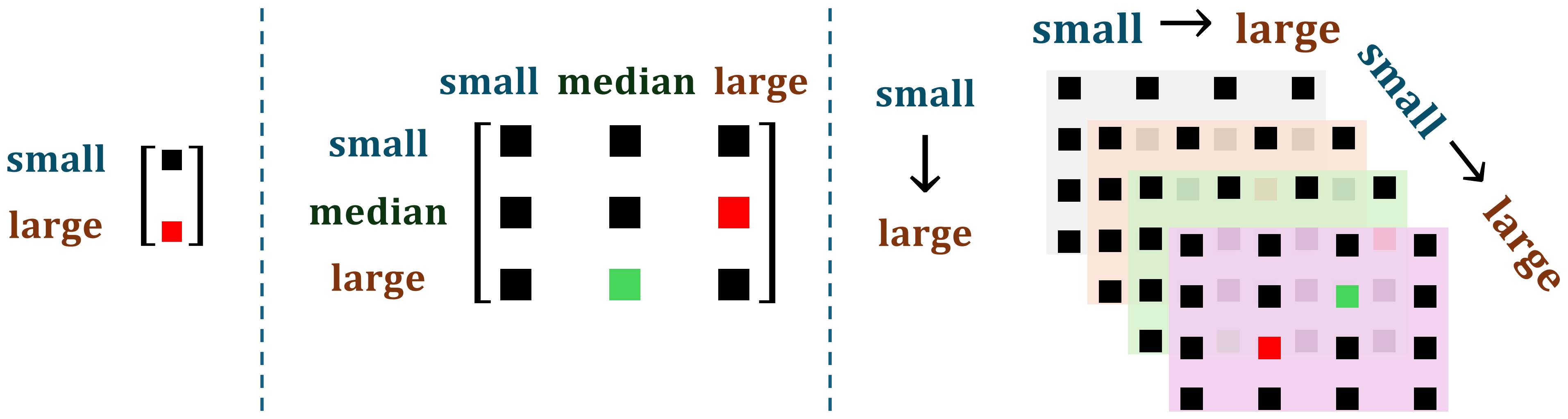
$k = 3$
R^2 vectors

$k = 4$

The SIS^{∞} problem

Linear dependence of
$\boldsymbol{R}$ vectors using coeffs $\pm\boldsymbol{B}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm\boldsymbol{B}/\boldsymbol{k}$



$\boldsymbol{k} = 2$
$\boldsymbol{R}$ vectors

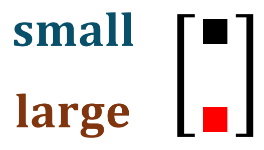
$\boldsymbol{k} = 3$
$\boldsymbol{R}^2$ vectors

$\boldsymbol{k} = 4$
$\boldsymbol{R}^3$ vectors

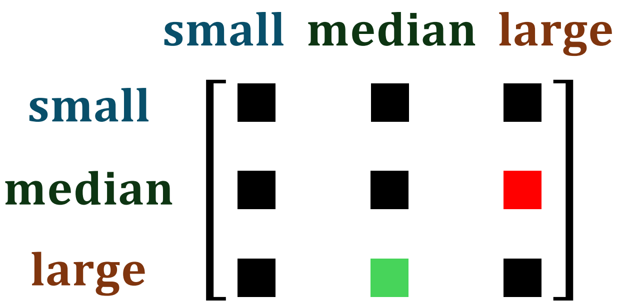
The SIS$^\infty$ problem

Linear dependence of
$\boldsymbol{R}$ vectors using coeffs $\pm\boldsymbol{B}$

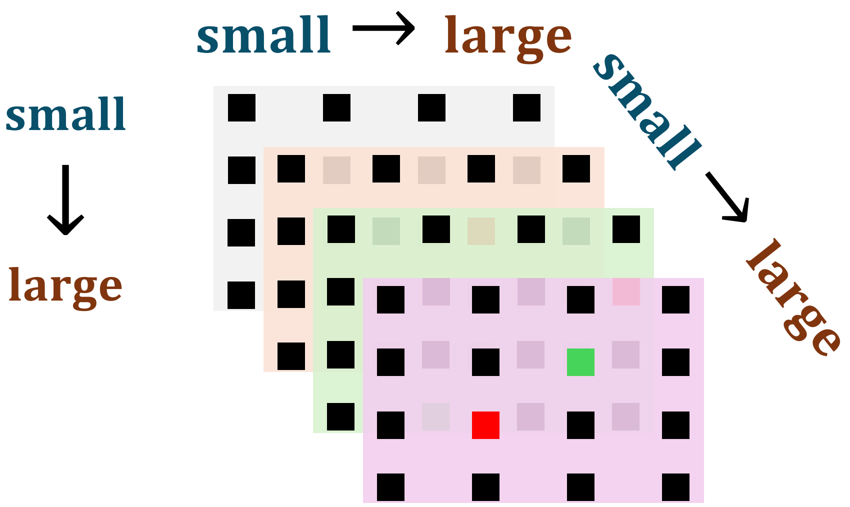
Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors
using coeffs in $\pm\boldsymbol{B}/\boldsymbol{k}$



$\boldsymbol{k = 2}$
$\boldsymbol{R}$ vectors



$\boldsymbol{k = 3}$
$\boldsymbol{R^2}$ vectors



$\boldsymbol{k = 4}$
$\boldsymbol{R^3}$ vectors

General $\boldsymbol{k}$

$(\boldsymbol{k-1})$-dim array

$\boldsymbol{R^{k-1}}$ vectors

Permutohedron

Algorithm overview

$\mathbf{F}_3^n$-Subset-Sum

Reducible vector

The SIS^∞ problem

Weight reduction

The $\boldsymbol{A}$-SIS problem

General reduction

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbf{F}_{\boldsymbol{p}}^{\boldsymbol{n}}$ and $\boldsymbol{A} \subseteq \mathbf{F}_{\boldsymbol{p}}$

Output: linear dependence using coeffs in $\boldsymbol{A}$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v_1}, \ldots, \boldsymbol{v_m} \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/3$

$$\begin{array}{cc} & \begin{array}{ccc} \text{small} & \text{median} & \text{large} \end{array} \\ \begin{array}{c} \text{small} \\ \text{median} \\ \text{large} \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

small $0 \sim B/3$
median $B/3 \sim 2B/3$
large $2B/3 \sim B$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{B}/\mathbf{3}$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$ is reducible

For example, if $\boldsymbol{c}$ is **large**, then

$$\begin{aligned}&\boldsymbol{c} \cdot \boldsymbol{u}\\ &= -\mathbf{small} \cdot \boldsymbol{d} + \mathbf{small} \cdot \boldsymbol{b}\\ &\quad +(\mathbf{median} - \mathbf{median}) \cdot \boldsymbol{e}\\ &\quad +(\boldsymbol{c} - \mathbf{large}) \cdot \boldsymbol{f} + (\mathbf{large} - \boldsymbol{c}) \cdot \boldsymbol{h}\end{aligned}$$

$$\begin{array}{cc} & \begin{array}{ccc}\mathbf{small} & \mathbf{median} & \mathbf{large}\end{array} \\ \begin{array}{c}\mathbf{small}\\ \mathbf{median}\\ \mathbf{large}\end{array} & \begin{bmatrix}\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t}\\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f}\\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i}\end{bmatrix} = \vec{0} \\ & = \vec{0}\end{array}$$

$\mathbf{small}$ $\mathbf{0} \sim \boldsymbol{B}/\mathbf{3}$
$\mathbf{median}$ $\boldsymbol{B}/\mathbf{3} \sim \mathbf{2}\boldsymbol{B}/\mathbf{3}$
$\mathbf{large}$ $\mathbf{2}\boldsymbol{B}/\mathbf{3} \sim \boldsymbol{B}$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \leq \boldsymbol{c} \leq \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\mathbf{3}$

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{small} & \mathbf{median} & \mathbf{large} \end{array} \\ \begin{array}{c} \mathbf{small} \\ \mathbf{median} \\ \mathbf{large} \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

small $\mathbf{0} \sim \boldsymbol{B}/\mathbf{3}$
median $\boldsymbol{B}/\mathbf{3} \sim \mathbf{2}\boldsymbol{B}/\mathbf{3}$
large $\mathbf{2}\boldsymbol{B}/\mathbf{3} \sim \boldsymbol{B}$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$ is reducible

For example, if $\boldsymbol{c}$ is **large**, then

$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= -\mathbf{small} \cdot \boldsymbol{d} + \mathbf{small} \cdot \boldsymbol{b} \\ &\quad +(\mathbf{median} - \mathbf{median}) \cdot \boldsymbol{e} \\ &\quad +(\boldsymbol{c} - \mathbf{large}) \cdot \boldsymbol{f} + (\mathbf{large} - \boldsymbol{c}) \cdot \boldsymbol{h} \end{aligned}$$

has coeffs in

- $\pm$**small**
- **median** $-$ **median**
- **large** $-$ **large**

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $-\boldsymbol{B} \le \boldsymbol{c} \le \boldsymbol{B}$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm \boldsymbol{B}/\mathbf{3}$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$ is reducible

$$\begin{array}{cc} & \begin{array}{ccc} \textbf{small} & \textbf{median} & \textbf{large} \end{array} \\ \begin{array}{c} \textbf{small} \\ \textbf{median} \\ \textbf{large} \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

small $\mathbf{0} \sim \boldsymbol{B}/\mathbf{3}$
median $\boldsymbol{B}/\mathbf{3} \sim \mathbf{2}\boldsymbol{B}/\mathbf{3}$
large $\mathbf{2}\boldsymbol{B}/\mathbf{3} \sim \boldsymbol{B}$

For example, if $\boldsymbol{c}$ is **large**, then

$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= -\mathbf{small} \cdot \boldsymbol{d} + \mathbf{small} \cdot \boldsymbol{b} \\ &\quad + (\mathbf{median} - \mathbf{median}) \cdot \boldsymbol{e} \\ &\quad + (\boldsymbol{c} - \mathbf{large}) \cdot \boldsymbol{f} + (\mathbf{large} - \boldsymbol{c}) \cdot \boldsymbol{h} \end{aligned}$$

has coeffs in

- $\pm$**small**
- **median** $-$ **median**
- **large** $-$ **large**

All in $\pm \boldsymbol{B}/\mathbf{3}$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $\boldsymbol{c} \in \pm(\mathbf{small} \cup \mathbf{median} \cup \mathbf{large})$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm$**small** $\cup$ (**median** $-$ **median**) $\cup$ (**large** $-$ **large**)

$$\begin{array}{c|c} & \begin{array}{ccc} \mathbf{small} & \mathbf{median} & \mathbf{large} \end{array} \\ \begin{array}{c} \mathbf{small} \\ \mathbf{median} \\ \mathbf{large} \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

small $\mathbf{0} \sim \boldsymbol{B}/\mathbf{3}$
median $\boldsymbol{B}/\mathbf{3} \sim \mathbf{2}\boldsymbol{B}/\mathbf{3}$
large $\mathbf{2}\boldsymbol{B}/\mathbf{3} \sim \boldsymbol{B}$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$ is reducible

For example, if $\boldsymbol{c}$ is **large**, then

$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= -\mathbf{small} \cdot \boldsymbol{d} + \mathbf{small} \cdot \boldsymbol{b} \\ &\quad + (\mathbf{median} - \mathbf{median}) \cdot \boldsymbol{e} \\ &\quad + (\boldsymbol{c} - \mathbf{large}) \cdot \boldsymbol{f} + (\mathbf{large} - \boldsymbol{c}) \cdot \boldsymbol{h} \end{aligned}$$

has coeffs in

$$\left.\begin{array}{l} \bullet\ \pm\text{small} \\ \bullet\ \text{median} - \text{median} \\ \bullet\ \text{large} - \text{large} \end{array}\right\} \text{All in } \pm\boldsymbol{B}/\mathbf{3}$$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $\boldsymbol{c} \in \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \boldsymbol{H}_2)$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup (\boldsymbol{H}_2 - \boldsymbol{H}_2)$

$$\begin{array}{cc} & \begin{array}{ccc} \boldsymbol{H}_0 & \boldsymbol{H}_1 & \boldsymbol{H}_2 \end{array} \\ \begin{array}{c} \boldsymbol{H}_0 \\ \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$ is reducible

For example, if $\boldsymbol{c}$ is in $\boldsymbol{H}_2$, then

$$\begin{aligned} &\boldsymbol{c} \cdot \boldsymbol{u} \\ &= -\boldsymbol{H}_0 \cdot \boldsymbol{d} + \boldsymbol{H}_0 \cdot \boldsymbol{b} \\ &\quad +(\boldsymbol{H}_1 - \boldsymbol{H}_1) \cdot \boldsymbol{e} \\ &\quad +(\boldsymbol{c} - \boldsymbol{H}_2) \cdot \boldsymbol{f} + (\boldsymbol{H}_2 - \boldsymbol{c}) \cdot \boldsymbol{h} \end{aligned}$$

has coeffs in

- $\pm\boldsymbol{H}_0$
- $\boldsymbol{H}_1 - \boldsymbol{H}_1$
- $\boldsymbol{H}_2 - \boldsymbol{H}_2$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).
$\boldsymbol{u}$ is reducible if for any $\boldsymbol{c} \in \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \boldsymbol{H}_2)$,
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$$\begin{array}{cc} & \begin{array}{ccc} \boldsymbol{H}_0 & \boldsymbol{H}_1 & \boldsymbol{H}_2 \end{array} \\ \begin{array}{c} \boldsymbol{H}_0 \\ \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \end{array} & \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{t} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} = \vec{0} \\ & = \vec{0} \end{array}$$

$\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}$ is reducible

For example, if $\boldsymbol{c}$ is in $\boldsymbol{H}_2$, then

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has coeffs in

- $\pm\boldsymbol{H}_0$
- $\boldsymbol{H}_1 - \boldsymbol{H}_1$
- $\boldsymbol{H}_2 - \boldsymbol{H}_2$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).

$\boldsymbol{u}$ is reducible if for any $\boldsymbol{c} \in \pm(\boldsymbol{H_0} \cup \boldsymbol{H_1} \cup \boldsymbol{H_2})$,
$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs
in $\pm\boldsymbol{H_0} \cup (\boldsymbol{H_1} - \boldsymbol{H_1}) \cup (\boldsymbol{H_2} - \boldsymbol{H_2})$

Theorem.

If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{A} = \pm(\boldsymbol{H_0} \cup \boldsymbol{H_1} \cup \boldsymbol{H_2})$,
then reducible vector exists given $\boldsymbol{R}^2$ vectors

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).

$\boldsymbol{u}$ is reducible if for any $\boldsymbol{c} \in \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \boldsymbol{H}_2)$,

$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs in $\pm\boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup (\boldsymbol{H}_2 - \boldsymbol{H}_2)$

Theorem.

If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{A} = \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \boldsymbol{H}_2)$,

then reducible vector exists given $\boldsymbol{R}^2$ vectors

Therefore $\boldsymbol{m} = \boldsymbol{R}^3$ suffices for $\boldsymbol{A} = \pm\boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup (\boldsymbol{H}_2 - \boldsymbol{H}_2)$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Def (reducible vector).

$\boldsymbol{u}$ is reducible if for any $\boldsymbol{c} \in \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \cdots \cup \boldsymbol{H}_k)$,

$\boldsymbol{c} \cdot \boldsymbol{u}$ is a linear comb of the given vectors using coeffs

in $\pm\boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup \cdots \cup (\boldsymbol{H}_k - \boldsymbol{H}_k)$

Theorem.

If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{A} = \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \cdots \cup \boldsymbol{H}_k)$,

then reducible vector exists given $\boldsymbol{R}^k$ vectors

Therefore $\boldsymbol{m} = \boldsymbol{R}^{k+1}$ suffices for $\boldsymbol{A} = \pm\boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup \cdots \cup (\boldsymbol{H}_k - \boldsymbol{H}_k)$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v_1, \ldots, v_m} \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Theorem.
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{A} = \pm(\boldsymbol{H_0} \cup \boldsymbol{H_1} \cup \cdots \cup \boldsymbol{H_k})$,
then $\boldsymbol{m} = \boldsymbol{R}^{\boldsymbol{k}+1}$ suffices for $\boldsymbol{A} = \pm \boldsymbol{H_0} \cup (\boldsymbol{H_1} - \boldsymbol{H_1}) \cup \cdots \cup (\boldsymbol{H_k} - \boldsymbol{H_k})$

The A-SIS problem

Input: $v_1, \ldots, v_m \in \mathbf{F}_p^n$ and $A \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in A

Theorem.
If $m = R$ suffices for $A = \pm(H_0 \cup H_1 \cup \cdots \cup H_k)$,
then $m = R^{k+1}$ suffices for $A = \pm H_0 \cup (H_1 - H_1) \cup \cdots \cup (H_k - H_k)$

Fact. If $A = \mathbf{F}_p$, then $m = n + 1$ suffices

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Theorem.
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{A} = \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \cdots \cup \boldsymbol{H}_k)$,
then $\boldsymbol{m} = \boldsymbol{R}^{k+1}$ suffices for $\boldsymbol{A} = \pm \boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup \cdots \cup (\boldsymbol{H}_k - \boldsymbol{H}_k)$

Fact. If $\boldsymbol{A} = \mathbf{F}_p$, then $\boldsymbol{m} = \boldsymbol{n} + \mathbf{1}$ suffices

Partition $\mathbf{F}_p = \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \cdots \cup \boldsymbol{H}_k)$
to obtain general $\boldsymbol{A}$

The A-SIS problem

Input: $v_1, \dots, v_m \in \mathbf{F}_p^n$ and $A \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in A

Theorem.
If $m = R$ suffices for $A = \pm(H_0 \cup H_1 \cup \cdots \cup H_k)$,
then $m = R^{k+1}$ suffices for $A = \pm H_0 \cup (H_1 - H_1) \cup \cdots \cup (H_k - H_k)$

Fact. If $A = \mathbf{F}_p$, then $m = n + 1$ suffices

Partition $\mathbf{F}_p = \pm(H_0 \cup H_1 \cup \cdots \cup H_k)$
to obtain general A

Example.
$p = 11$ and $\mathbf{F}_p = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$
$H_0 = \{0, 3, 4, 5\}$, $H_1 = \{1\}$, and $H_2 = \{2\}$

The $\boldsymbol{A}$-SIS problem

Input: $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbf{F}_p^n$ and $\boldsymbol{A} \subseteq \mathbf{F}_p$

Output: linear dependence using coeffs in $\boldsymbol{A}$

Theorem.
If $\boldsymbol{m} = \boldsymbol{R}$ suffices for $\boldsymbol{A} = \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \cdots \cup \boldsymbol{H}_k)$,
then $\boldsymbol{m} = \boldsymbol{R}^{k+1}$ suffices for $\boldsymbol{A} = \pm\boldsymbol{H}_0 \cup (\boldsymbol{H}_1 - \boldsymbol{H}_1) \cup \cdots \cup (\boldsymbol{H}_k - \boldsymbol{H}_k)$

Fact. If $\boldsymbol{A} = \mathbf{F}_p$, then $\boldsymbol{m} = \boldsymbol{n} + \mathbf{1}$ suffices

Partition $\mathbf{F}_p = \pm(\boldsymbol{H}_0 \cup \boldsymbol{H}_1 \cup \cdots \cup \boldsymbol{H}_k)$
to obtain general $\boldsymbol{A}$

Example.
$\boldsymbol{p} = \mathbf{11}$ and $\mathbf{F}_p = \{\mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{4}, \pm\mathbf{5}\}$
$\boldsymbol{H}_0 = \{\mathbf{0}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, $\boldsymbol{H}_1 = \{\mathbf{1}\}$, and $\boldsymbol{H}_2 = \{\mathbf{2}\}$

Then $\boldsymbol{A} = \{\mathbf{0}, \pm\mathbf{3}, \pm\mathbf{4}, \pm\mathbf{5}\}$
And $\boldsymbol{m} \approx \boldsymbol{n}^3$ suffices

Summary

Classical algorithms matching/improving previous quantum algorithms on Short-Integer-Solution-related problem

- $\mathbf{F}_3^n$-Subset-Sum
- SIS^∞
- $\boldsymbol{A}$-SIS

Summary

Classical algorithms matching/improving previous quantum algorithms on Short-Integer-Solution-related problem

- $\mathbf{F}_3^n$-Subset-Sum
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No quantum speedup for these problems anymore

Summary

Classical algorithms matching/improving previous quantum algorithms on Short-Integer-Solution-related problem

$$\left.\begin{array}{l}\mathbf{F}_3^n\text{-Subset-Sum} \\ \mathrm{SIS}^\infty \\ \boldsymbol{A}\text{-SIS}\end{array}\right\}\ \text{We do not break any crypto assumption}$$

No quantum speedup for these problems anymore

Summary

Classical algorithms matching/improving previous quantum algorithms on Short-Integer-Solution-related problem

$\mathbf{F}_3^n$-Subset-Sum
SIS^∞
$\boldsymbol{A}$-SIS

We do not break any crypto assumption

No quantum speedup for these problems anymore

Candidate quantum speedup still exists for quantum algorithms similarly based on Regev's reduction

Summary

Classical algorithms matching/improving previous quantum algorithms on Short-Integer-Solution-related problem

- $\mathbf{F}_3^n$-Subset-Sum
- SIS^∞
- $\boldsymbol{A}$-SIS

We do not break any crypto assumption

No quantum speedup for these problems anymore

Candidate quantum speedup still exists for quantum algorithms similarly based on Regev's reduction

Thank you!